FRAMES GENERATED BY ACTIONS OF LOCALLY COMPACT GROUPS

by

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DISSERTATION ABSTRACT

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Title: Frames Generated by Actions of Locally Compact Groups

Let G be a second countable, locally compact group which is either compact or abelian, and let ρ be a unitary representation of G on a separable Hilbert space $\mathcal{H}_\rho$. We examine frames of the form $\{\rho(x)f_j : x \in G, j \in I\}$ for families $\{f_j\}_{j\in I}$ in $\mathcal{H}_\rho$. In particular, we give necessary and sufficient conditions for the joint orbit of a family of vectors in $\mathcal{H}_\rho$ to form a continuous frame.

We pay special attention to this problem in the setting of shift invariance. In other words, we fix a larger second countable locally compact group $\Gamma \supseteq G$ containing G as a closed subgroup, and we let ρ be the action of G on $L^2(\Gamma)$ by left translation. In both the compact and the abelian settings, we introduce notions of Zak transforms on $L^2(\Gamma)$ which simplify the analysis of group frames. Meanwhile, we run a parallel program that uses the Zak transform to classify closed subspaces of $L^2(\Gamma)$ which are invariant under left translation by G. The two projects give compatible outcomes.

This dissertation contains previously published material.

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In honor of my grandparents: Leo and Leona Feldhaus, and
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CHAPTER I

INTRODUCTION

This chapter contains previously published material. In fact, this entire dissertation is a combination of two previously existing articles. Chapters II and III were previously published as [42], while Chapter IV will be published as [43]. This introduction contains portions of the introductions in [42, 43]. The sole exception is Section 3.4, which is appearing here for the first time. The reader making a citation is encouraged to refer to the original sources rather than to this compilation.

This dissertation focuses on the interplay between frame theory and representations of locally compact groups. Broadly speaking, we are interested in two related questions about a unitary representation π of a locally compact group G:

(Q1) What are the invariant subspaces of π?

and

(Q2) For which families $\mathscr{A}$ of vectors in the representation space is the orbit $\{\pi(x)f \colon x \in G, f \in \mathscr{A}\}$ a frame?

These questions are related in the following way. In general, the vectors $\{\pi(x)f \colon x \in G, f \in \mathscr{A}\}$ may not span the entire representation space, in which case they can only form a frame for their closed linear span. That span is precisely the invariant subspace generated by $\mathscr{A}$. In most situations, therefore, one cannot hope to answer (Q2) without first addressing (Q1).

An overarching theme of this document is the notion that frame theory and representation theory share deep connections. This goes far beyond the prominence of reproducing systems associated with group actions. On the face of things, for instance, (Q1) is a question about representation theory, and (Q2) is a question about frame theory. In Chapter IV, however, we will use tools from frame theory to answer (Q1), and from representation theory to answer (Q2). In fact, many of the standard tools of frame theory give vital information about the structure of representations. For instance, another part of Chapter IV introduces an analogue of the *bracket map*, which found its first use in the study of multiresolution analysis [46]. It will turn out that our version of the bracket carries information about the isotypical components of a representation and the multiplicities of irreducibles, and in many cases can be used to test a purported cyclic vector. We will also give a complete description of the invariant subspaces of an arbitrary representation of a compact group, and explain how to use one irreducible decomposition to classify all such decompositions. The main tool for both of these applications is essentially the analysis operator, which belongs firmly in the toolbox labeled "frame theory".

1.1. Shift-invariant spaces

Our questions (Q1) and (Q2) have been explored most thoroughly in the context of shift-invariant spaces, where they made some of their first appearances. From our perspective, shift-invariant spaces come from the action of $G = \mathbb{Z}^n$ on $L^2(\mathbb{R}^n)$ by integer shifts $T_k \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$,

$$(T_k f)(x) = f(x-k) \qquad \text{for } k \in \mathbb{Z}^n, f \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n.$$

The frames associated with this representation have the form $\{T_k f_i \colon k \in \mathbb{Z}^n, i \in I\}$. They appear most prominently in Gabor systems and multiresolution analysis, with important applications for wavelets, spline systems, and approximation theory.

The invariant subspaces of this representation are called *shift-invariant*, or SI. In other words, an SI-space is a closed subspace $V \subseteq L^2(\mathbb{R}^n)$ such that $T_k V = V$ for every $k \in \mathbb{Z}^n$. In the 1960s, Helson [37] and Srinivasan [57] used range functions to classify SI-spaces, under the equivalent guise of "doubly invariant spaces". This mask was uncovered by de Boor, DeVore, and Ron [22] in 1994, through the lens of the *fiberization* operator

$$\mathcal{T}\colon L^2(\mathbb{R}^n) \to L^2([0,1]^n; \ell^2(\mathbb{Z}^n))$$

given by

$$(\mathcal{T}f)(x) = \left(\hat{f}(x+k)\right)_{k\in\mathbb{Z}^n} \qquad \text{for } f \in L^2(\mathbb{R}^n) \text{ and } x \in [0,1]^n.$$

In other words, $(\mathcal{T}f)(x)$ gives the values of $\hat{f}$ on the coset $x + \mathbb{Z}^n$, in the form of a sequence which happens to belong to $\ell^2(\mathbb{Z}^n)$. Fiberization was the key to almost all research on SI-spaces before 2014, when the paper [42] behind Chapters II–III and the concurrent work of Hernández and his colleagues [10] introduced the Zak transform as an alternative. A landmark 2000 paper of Bownik [11] built on [22] to give an explicit reformulation of Helson's classification of SI-spaces in terms of range functions and the fiberization operator. More importantly, he tied this classification into a broader theory of frames generated by integer shifts in $L^2(\mathbb{R}^n)$. In other words, he answered (Q1) and (Q2) for integer shifts in $L^2(\mathbb{R}^n)$, and he combined his answers into a single, tidy package.

What followed was a mass effort to extract and generalize the group-theoretic underpinnings of [11], replacing $\mathbb{R}^n$ with a general locally compact abelian (LCA) group Γ, and $\mathbb{Z}^n$ with a closed subgroup $G \subseteq \Gamma$. Here as in the classical setting,

G acts on $L^2(\Gamma)$ by translation. The problem is to answer (Q1) and (Q2) for this representation. Following the blueprint of [11], a variety of authors [13, 14, 49] worked toward a solution by first imposing and then carefully removing topological hypotheses on the subgroup $G \subseteq \Gamma$.

Before the material in this dissertation appeared, the state of the art required that Γ/G be compact. This was a major obstruction, because it precluded important examples like the p-adics $\mathbb{Z}_p \subseteq \mathbb{Q}_p$, as well as basic ones like a lattice $\mathbb{Z}^m \subseteq \mathbb{R}^n$ of less than full rank. Without this technical assumption, however, it was not clear how to define the fiberization operator, or even whether such an operator existed. Our solution in Chapters II and III overcomes this barrier by exhibiting a very general measure space isomorphism $\Gamma \cong G \times \Gamma/G$ which makes fiberization almost trivial. This leads to a complete characterization of shift-invariance in the abelian setting. However, the larger contribution of Chapter III is to give an additional characterization of SI-spaces and frames of translates that replaces fiberization with a version of the Zak transform. The Zak transform shifts the burden of Fourier analysis from the large group Γ to the small group G, thereby eliminating the need for Γ to be abelian. This allows for much more sophisticated examples, like the non-normal copy of $\mathbb{R}$ in the $ax + b$ group. At the same time, it produces a description of shift-invariant spaces in $L^2(\mathbb{R}^n)$ more amenable to the theory of Gabor systems, where the Zak transform plays a prominent role.

In Chapter IV, we push these ideas much further in the nonabelian direction, even removing the assumption that the subgroup G be abelian. Instead, we require G to be compact. Once again leveraging the measure space isomorphism from Chapter II, we develop an operator-valued version of the Zak transform for $L^2(\Gamma)$. This forms the basis for a range-function characterization of shift-invariant spaces and frames of translates in the nonabelian setting. Along with the characteristically different papers [9, 19], this forms one of the first successful excursions into the world of shift invariance without commutativity assumptions.

1.2. Group frames

Frames of the form in (Q2) are called *group frames*. The vectors $f \in \mathscr{A}$ are called *generators*. If $\operatorname{card}(\mathscr{A}) = 1$, the frame is said to have a *single generator*; otherwise, it has *multiple generators*. Group frames are reproducing systems made by leveraging the natural symmetries of a Hilbert space. This often results in a certain austere beauty, and eases the surrounding calculations. In particular, the reproducing properties of group frames can be greatly simplified by the aid of the representation. When there is a single generator f, for instance, the frame operator S lies in the commutant of π. This means that when one wants to reproduce a vector g with the formula

$$g = \int_G \langle g, S^{-1}\pi(x)f\rangle \pi(x)f \, dx,$$

there is no need to actually find the canonical dual $\{S^{-1}\pi(x)f : x \in G\}$. It suffices to compute $S^{-1}f$ and then observe that

$$\langle g, S^{-1}\pi(x)f\rangle = \langle \pi(x^{-1})g, S^{-1}f\rangle.$$

In practice, this means we just have to make a one-time payment to find the inverse image of a single vector, and after that we can reproduce vectors to our hearts' content until the end of time.

Group frames are at least as old as modern frame theory; in a sense, they are even older. A 1985 paper of Grossmann, Morlet, and Paul [33] describes a method for constructing reproducing systems using irreducible representations of locally compact groups. In modern language, we would call those systems "tight frames". The landmark 1986 paper of Daubechies, Grossmann, and Meyer [20], which revived modern interest in frames, leaned heavily on [33], and described Gabor systems and wavelets in terms of unitary representations of the Weyl-Heisenberg and $ax+b$ groups, respectively.

One of the first systematic treatments of general group frames appeared in the 2000 monograph of Han and Larson [36], which established many of their operator-theoretic properties. Actually *constructing* examples in the sense of (Q2), however, was another issue. After wavelets and Gabor systems, some of the first examples were harmonic frames, which are made by extracting rows from a discrete Fourier transform (DFT) matrix. Harmonic frames are associated with actions of abelian groups, and were studied by several independent authors in the early 2000s [16, 24, 29, 30, 58]. The nonabelian case has proved more difficult. It has long been known that when G is finite and the representation π is irreducible, any single nonzero vector generates a tight frame in the sense of (Q2). The first progress on *reducible* representations of finite nonabelian groups was made by Vale and Waldron [59] in 2008. By decomposing the representation as a direct sum of irreducibles, they established a neat condition for a single vector to generate a tight frame. This was the state of the art in nonabelian group frames for quite some time. It was not clear how to extend this result to allow frames with unequal bounds or, more annoyingly, multiple generating vectors. As recently as 2013, Waldron [61] described characterizing frames with multiple generators as a major unsolved problem in this area.

In the specific context of shift-invariant spaces, however, frames with multiple generators have been routinely characterized since at least the work of Ron and Shen

[55] in 1995. Borrowing from this tradition, our work in Chapter IV uses shift-invariant spaces as a kind of all-purpose machine for the theory of group frames, resulting in a characterization of general group frames in the compact setting—including frames with multiple generators. This takes the form of a very general duality theorem. Around the same time as the paper behind Chapter IV was released, Vale and Waldron [60] independently found a solution of the multiple generator problem in the special case of finite groups and tight frames. As it turned out, their main result was a special case of the duality theorem in Chapter IV.

1.3. Bracket analysis

Another line of research on group frames stems from the *bracket map* introduced by Jia and Micchelli [46] in 1991 for the analysis of principal shift-invariant spaces. In 2010, Hernández, Šikić, Weiss, and Wilson [39] developed a notion of the bracket map for a special class of representations of LCA groups. Representations that admit a bracket are called "dual integrable". Weiss and his collaborators used the bracket to describe single generator frames made by dual integrable representations of *discrete* abelian groups. In [42], I found a way to connect this problem with shift-invariance on LCA groups, which led to an extension of the results in [39] for general (not necessarily discrete) LCA groups and, significantly, multiple generators.

Bracket analysis is user-friendly in the sense that it does not require any understanding of an irreducible decomposition for the representation. On the other hand, it may be extremely difficult to actually *compute* a bracket in practice, even when abstract machinery guarantees its existence. Our work in Chapter IV solves this problem in the case of compact groups. Expanding on the ideas in [39], we give an operator-valued version of the bracket that works for *any* representation of a compact group. Importantly, the bracket can be computed *directly from the representation itself*, still without requiring knowledge of the representation's structure. On the contrary, a major application of the bracket lies in its ability to *compute* that structure from first principles. This is an important example of frame theory paying down its debt to representation theory. The expanded version of the bracket still works for group frames with a single generator. In fact, the whole package of bracket analysis is flexible enough to produce a complete classification of single generator frames associated with actions of compact groups. This, in turn, leads to a number of new examples, including a generalization of the harmonic frame construction for nonabelian groups.

1.4. Overview

This document is organized as follows. Chapter II contains top-level machinery necessary for the sequel. In Section 2.1, we investigate a common source of frames and Riesz bases in a measure-theoretic setting. Let X be a measure space, and let $\mathcal{H}$ be a separable Hilbert space. Any two functions $f\colon X \to \mathbb{C}$ and $\varphi\colon X \to \mathcal{H}$ can be multiplied pointwise, with product $(f\varphi)(x) = f(x)\varphi(x)$. Given a basis-like set $\mathcal{D} \subseteq L^2(X)$ and a family $\mathscr{A} \subseteq L^2(X;\mathcal{H})$, we give conditions under which $\{f\varphi\colon f \in \mathcal{D}, \varphi \in \mathscr{A}\}$ forms a continuous frame or a Riesz sequence in $L^2(X;\mathcal{H})$. The main results here are Theorems 2.1.3 and 2.1.10, relating frame and Riesz sequence conditions in $L^2(X;\mathcal{H})$ to the corresponding pointwise conditions in $\mathcal{H}$. This section is meant as a companion to the recent work by Bownik and Ross [13, §2] on range functions and multiplicative invariance in the measure-theoretic setting.

In Section 2.2, we develop a version of Weil's formula for right cosets. Given a closed subgroup $\Gamma \subseteq G$, we produce a measure on the space $\Gamma\backslash G$ that allows for a measure space isomorphism $G \cong \Gamma \times \Gamma \backslash G$. This isomorphism is the key to our development of the abstract Zak transform and fiberization map in Section 3.1. There we give a wide variety of examples, and describe connections between the Zak transform and the fiberization map in the abelian setting.

Actions of abelian groups are the focus of Chapter III. To begin, we fix an abelian subgroup $H \subseteq G$, and consider its action on $L^2(G)$ by left translations $L_\xi \colon L^2(G) \to L^2(G)$:

$$(L_\xi f)(x) = f(\xi^{-1}x) \qquad \text{for } f \in L^2(G),\ \xi \in H, \text{ and } x \in G.$$

A closed subspace $V \subseteq L^2(G)$ satisfying the condition $L_\xi V = V$ for all $\xi \in H$ will be called *H-translation invariant*, or H-TI. In Section 3.2, we give our main results classifying H-TI spaces in terms of fiberization and/or the Zak transform. Here we also describe conditions under which the left H-translates of a family of functions in $L^2(G)$ form a continuous frame or a Riesz sequence. At the end of this section, we analyze translation/modulation-invariant spaces under critical sampling in the abelian setting.

In Section 3.3, we consider the related problem of invariant subspaces for dual integrable representations of locally compact abelian (LCA) groups, introduced by Hernández, Šikić, Weiss, and Wilson in [40]. We show that dual integrable representations are precisely those gotten from the translation action of an abelian subgroup, as in Section 3.2. We then give a range function classification of invariant subspaces. Moreover, we explain when the orbit of a family of vectors produces a continuous frame. If the group is discrete, we do the same for Riesz sequences. Our results generalize those in [40], which treated discrete LCA groups and cyclic subspaces. Section 3.4, an addendum to the original article [42], makes the connection between group frames and shift invariant spaces explicit, and serves as a bridge between the two articles that make up this dissertation.

Chapter IV deals with actions of compact groups. The first part, Sections 4.1–4.3, investigates questions (Q1) and (Q2) for actions of compact groups by translation. Let G be a second countable locally compact group, and let $K \subseteq G$ be a compact subgroup. The purpose of these sections is to describe the structure of closed subspaces of $L^2(G)$ which are invariant under left translation by K. We call these spaces K-*invariant*. Our first major development occurs in Section 4.1, where we introduce an operator-valued analogue of the Zak transform, generalizing a classical construction of Weil [62, 63] and Gelfand [28]. It forms the basis for much of our subsequent analysis. In Section 4.2, we use range functions to classify K-invariant subspaces of $L^2(G)$, and explore this correspondence in depth. This line of thinking comes to a culmination in Section 4.3, where we give precise conditions for a family of functions in $L^2(G)$ to generate a frame via left translation by K.

The second part of the paper, Sections 4.4 and 4.5, describes a symbolic calculus for the analysis of representations of compact groups. We introduce an operator-valued version of the bracket map first developed for the study of principal shift-invariant spaces by Jia and Micchelli [46], and subsequently generalized for actions of locally compact abelian (LCA) groups by Weiss and his collaborators [39], then by a variety of authors in other settings [7, 8, 9]. Our main result, Theorem 4.4.3, gives the frame properties of the orbit of a cyclic vector in terms of the eigenvalues of the bracket. We develop basic properties of the bracket in Section 4.4. Several of these show the bracket carries vital information about the structure of the representation itself. Section 4.5 contains a host of applications: classification of group frames with a single generator, block diagonalization of the Gramian operator, disjointness properties, and several new examples of frames, including a generalization of harmonic frames for nonabelian groups.

The third part, Section 4.6, is dedicated to group frames with multiple generators. Here we mimic the program of Sections 4.1–4.3 for an arbitrary representation ρ of a compact group K, assuming only that we know how to decompose ρ as a direct sum of irreducible subrepresentations. We classify the invariant subspaces of ρ using range functions and a sort of analysis operator, then describe every possible decomposition of the representation space as a direct sum of irreducible invariant subspaces. The capstone of this section is the duality result in Theorem 4.6.1, which answers (Q2) for arbitrary representations of compact groups. Our result unifies classical duality of frames and Riesz sequences with, among other things, the pioneering work of Vale and Waldron [58, 59, 60], and the well-known result that the orbit of a nonzero vector under an irreducible representation of K always forms a tight frame. We hope that this theorem, and many of the other ideas in this document, will give some clues for subsequent research on group frames.

CHAPTER II

ABSTRACT MACHINERY

This chapter was previously published as [42, §2–3].

2.1. Frames and Riesz bases in $L^2(X;\mathcal{H})$

Let $\mathscr{A}$ be a countable family of functions in $L^2(\mathbb{R}^n)$, and let

$$E(\mathscr{A}) = \{T_k f \colon k \in \mathbb{Z}^n, f \in \mathscr{A}\},$$

where $T_k f(y) = f(y-k)$. In [11], Bownik studied $E(\mathscr{A})$ through the fiberization map $\mathcal{T} \colon L^2(\mathbb{R}^n) \to L^2([0,1)^n; l^2(\mathbb{Z}^n))$ given by

$$\mathcal{T} f(x) = (\hat{f}(x+k))_{k\in\mathbb{Z}^n} \qquad \text{for } x \in [0,1)^n,$$

where the Fourier transform on $\mathbb{R}^n$ is normalized by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{-2\pi i \xi\cdot x}\, d\xi \qquad \text{for } x \in \mathbb{R}^n \text{ and } f \in L^1(\mathbb{R}^n).$$

The utility of $\mathcal{T}$ comes from the intertwining relation

$$\mathcal{T}(T_k f)(x) = e^{2\pi i k\cdot x}\mathcal{T} f(x) \quad \text{for all } k \in \mathbb{Z}^n,$$

so that integer shifts in $L^2(\mathbb{R}^n)$ become modulations by an orthonormal basis of $L^2([0,1)^n)$ in $L^2([0,1)^n; l^2(\mathbb{Z}^n))$. Taking advantage of this correspondence, Bownik gave sufficient and necessary conditions for $E(\mathscr{A})$ to form a frame or a Riesz basis for its closed linear span. Later, Cabrelli and Paternostro [14] and Kamyabi Gol and Raisi Tousi [49] generalized this method to the setting of a second countable LCA group G with a closed discrete subgroup H such that G/H is compact. Bownik and Ross [13] have gone even further by removing the hypothesis that H be discrete, replacing frames with so-called continuous frames. Each of these papers achieves its goal by transforming $L^2(G)$ into a space $L^2(X;\mathcal{H})$, with X a measure space and $\mathcal{H}$ a Hilbert space, in such a way that translations by the subgroup $H \subseteq G$ become modulations by a nice family of functions in $L^\infty(X)$. In this section, we consider the latter situation more generally. Namely, for a countable family $\mathscr{A} \subseteq L^2(X;\mathcal{H})$ and a basis-like family $\mathcal{D}$ of functions on X, we investigate conditions under which the family of functions $\{g\varphi\colon g \in \mathcal{D},\ \varphi \in \mathscr{A}\}$ given by

$$(g\varphi)(x) = g(x) \cdot \varphi(x) \qquad \text{for } x \in X$$

form a continuous frame or a Riesz basis for their closed linear span. This work is complementary to a recent publication by Bownik and Ross [13, §2] extending Helson's theory of multiplicative invariance [38]. We now describe their main results.

Definition 2.1.1. Let (X,μ) be a measure space. A *determining set* for $L^1(X)$ is a family of functions $\mathcal{D} \subseteq L^\infty(X)$ with the property that, for all $f \in L^1(X)$,

$$\int_X f(x)g(x)\,d\mu(x) = 0 \quad \text{for all } g \in \mathcal{D} \implies f = 0.$$

Given a separable Hilbert space $\mathcal{H}$, a closed subspace $M \subseteq L^2(X;\mathcal{H})$ is said to be *$\mathcal{D}$-multiplication invariant*, or $\mathcal{D}$-MI, if for every $g \in \mathcal{D}$ and every $\varphi \in M$, the function

$(g\varphi)(x) = g(x)\varphi(x)$ also belongs to M. Given a family $\mathscr{A}$ in $L^2(X;\mathcal{H})$, we define

$$S_{\mathcal{D}}(\mathscr{A}) = \overline{\operatorname{span}}\{g\varphi\colon g \in \mathcal{D}, \varphi \in \mathscr{A}\}$$

for the $\mathcal{D}$-MI space it generates, and

$$E_{\mathcal{D}}(\mathscr{A}) = \{g\varphi\colon g \in \mathcal{D}, \varphi \in \mathscr{A}\}.$$

We will consider $E_{\mathcal{D}}(\mathscr{A})$ as a set with multiplicities.

A *range function* is a mapping

$$J\colon X \to \{\text{closed subspaces of } \mathcal{H}\}.$$

For a range function J and $x \in X$, we write $P_J(x)\colon \mathcal{H} \to \mathcal{H}$ for the orthogonal projection onto $J(x)$. We say that J is a *measurable* range function if, for each $(u, v) \in \mathcal{H} \times \mathcal{H}$, the function $x \mapsto \langle P_J(x)u, v\rangle$ is measurable on X.

Range functions have a long history in the classification of invariant subspaces, dating at least as far back as Helson [37] and Srinivasan [57] in 1964. More recently, Bownik [11] used range functions to classify shift invariant subspaces of $L^2(\mathbb{R}^n)$. This program was continued in increasing generality by Cabrelli and Paternostro [14], Kamyabi Gol and Raisi Tousi [49], Currey, Mayeli, and Oussa [19], and Bownik and Ross [13]. Our results in Sections 3.2 and 3.3 continue this line of research.

The proposition below is a slight modification of Theorem 2.4 in [13]. See also Srinivasan [57, Theorem 3] and Helson [37, Lecture VI, Theorem 8] and [38, Ch. 1, §3, Theorem 1].

Proposition 2.1.2. *Let (X, μ) be a σ-finite measure space, let $\mathcal{D}$ be a determining set for $L^1(X)$, and let $\mathcal{H}$ be a separable Hilbert space.*

(i) If $J \colon X \to \{\text{closed subspaces of } \mathcal{H}\}$ is a range function, then

$$M_J = \{\varphi \in L^2(X; \mathcal{H}) \colon \varphi(x) \in J(x) \text{ for a.e. } x \in X\}$$

is a closed $\mathcal{D}$-MI subspace of $L^2(X; \mathcal{H})$.

(ii) The correspondence $J \mapsto M_J$ is a bijection between measurable range functions and closed $\mathcal{D}$-MI subspaces of $L^2(X; \mathcal{H})$, provided we identify range functions that agree a.e. on X.

(iii) Let $\mathscr{A}$ be a family of functions in $L^2(X; \mathcal{H})$, let $\mathscr{A}_0 \subseteq \mathscr{A}$ be a countable dense subset, and let J be the range function defined almost everywhere by

$$J(x) = \overline{\operatorname{span}}\{\varphi(x) \colon \varphi \in \mathscr{A}_0\}. \tag{2.1}$$

Then

$$M_J = S_{\mathcal{D}}(\mathscr{A}_0) = S_{\mathcal{D}}(\mathscr{A}). \tag{2.2}$$

Proof. In [13], Proposition 2.1 and Theorem 2.4 prove everything except the fact that $S_{\mathcal{D}}(\mathscr{A}_0) = S_{\mathcal{D}}(\mathscr{A})$. One inclusion in this equality is obvious. For the other, let $\varphi \in \mathscr{A}$ be arbitrary, and let $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence in $\mathscr{A}_0$ with $\varphi_k \to \varphi$. By passing to a subsequence if necessary, we may assume that $\varphi_k(x) \to \varphi(x)$ a.e. Since J maps X into the set of closed subspaces, $\varphi(x) \in J(x)$ for a.e. $x \in X$. In other words, $\varphi \in M_J = S_{\mathcal{D}}(\mathscr{A}_0)$. Since this holds for every $\varphi \in \mathscr{A}$, $S_{\mathcal{D}}(\mathscr{A}) \subseteq S_{\mathcal{D}}(\mathscr{A}_0)$. $\square$

2.11. Riesz sequences

We remind the reader that a countable family $(u_i)_{i\in I}$ of vectors in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there are constants $0 < A \leq B < \infty$ such that for all $(c_i)_{i\in I} \in l^2(I)$ with $c_i \neq 0$ for only finitely many i,

$$A \sum_{i\in I} |c_i|^2 \leq \left\| \sum_{i\in I} c_i u_i \right\|^2 \leq B \sum_{i\in I} |c_i|^2. \tag{2.3}$$

The constants A and B are called *bounds*. When a Riesz sequence spans a dense subspace of $\mathcal{H}$, it is called a *Riesz basis*.

The following theorem is an abstract version of [11, Theorem 2.3(ii)]. Our proof is a modification of the argument given there. See also [14, Theorem 4.3] and [13, Theorem 5.1].

Theorem 2.1.3. *Let (X,μ) be a measure space with $\mu(X) < \infty$, and let $\mathcal{H}$ be a Hilbert space. For a countable family $\mathscr{A}$ in $L^2(X;\mathcal{H})$ and constants $0 < A \leq B < \infty$, the following are equivalent:*

(i) For some orthonormal basis $\mathcal{D}$ of $L^2(X)$, $\{g\varphi\colon g \in \mathcal{D}$ and $\varphi \in \mathscr{A}\}$ is a Riesz sequence in $L^2(X;\mathcal{H})$ with bounds A and B.

(ii) For any Riesz sequence $(g_i)_{i\in I}$ in $L^2(X)$ with bounds a and b, $\{g_i\varphi\colon i \in I$ and $\varphi \in \mathscr{A}\}$ is a Riesz sequence in $L^2(X;\mathcal{H})$ with bounds aA and bB.

(iii) For any family $(f_\varphi)_{\varphi\in\mathscr{A}}$ in $L^2(X)$ having $f_\varphi \neq 0$ for only finitely many φ,

$$A \sum_{\varphi\in\mathscr{A}} \int_X |f_\varphi(x)|^2 \, d\mu(x) \leq \int_X \left\| \sum_{\varphi\in\mathscr{A}} f_\varphi(x)\varphi(x) \right\|^2 d\mu(x) \leq B \sum_{\varphi\in\mathscr{A}} \int_X |f_\varphi(x)|^2 \, d\mu(x). \tag{2.4}$$

(iv) For any family $(f_\varphi)_{\varphi\in\mathscr{A}}$ in $L^2(X)$ with $\sum_{\varphi\in\mathscr{A}} \|f_\varphi\|^2_{L^2(X)} < \infty$, (2.4) holds.

(v) For almost every $x \in X$, $\{\varphi(x) \colon \varphi \in \mathscr{A}\}$ is a Riesz sequence in $\mathcal{H}$ with bounds A and B.

Condition (iii) can be read as a strictly stronger version of the usual definition of Riesz sequence, where the coefficient sequence $(c_\varphi)_{\varphi \in \mathscr{A}}$ in $l^2(\mathscr{A})$ has been replaced with a function sequence $(f_\varphi)_{\varphi \in \mathscr{A}}$ in $l^2(\mathscr{A}; L^2(X))$.

Proof of Theorem 2.1.3. (iii) $\implies$ (v). Suppose there is a set $Y \subseteq X$ of positive measure such that, for each $x \in Y$, $\{\varphi(x) \colon \varphi \in \mathscr{A}\}$ is *not* a Riesz sequence in $\mathcal{H}$ because it fails the upper bound of (2.3). We'll show that (2.4) fails in the upper bound. Let $\{d_m\}_{m=1}^\infty$ be a dense subset of $l^2(\mathscr{A})$ such that each $d_m = (d_{m,\varphi})_{\varphi \in \mathscr{A}}$ has only finitely many nonzero entries. For each $m, n \in \mathbb{N}$, put

$$E_{m,n} = \{x \in X \colon \left\| \sum_{\varphi \in \mathscr{A}} d_{m,\varphi} \varphi(x) \right\|^2 > \left(B + \frac{1}{n}\right) \sum_{\varphi \in \mathscr{A}} |d_{m,\varphi}|^2\},$$

which is well-defined up to a set of measure zero. If $x \notin \bigcup_{m,n=1}^\infty E_{m,n}$, then

$$\left\| \sum_{\varphi \in \mathscr{A}} d_{m,\varphi} \varphi(x) \right\|^2 \leq B \sum_{\varphi \in \mathscr{A}} |d_{m,\varphi}|^2 \quad \text{for all } m,$$

so $\{\varphi(x) : \varphi \in \mathscr{A}\}$ satisfies the upper bound of the Riesz condition with bound B. Consequently, one of the sets $E_{m,n}$ has $\mu(E_{m,n}) > 0$, and for this m and n we define a family of functions $(f_\varphi)_{\varphi \in \mathscr{A}} \subseteq L^2(X)$ by the formula $f_\varphi(x) = d_{m,\varphi} \mathbf{1}_{E_{m,n}}(x)$. Only finitely many of these functions are nonzero, yet

$$\int_X \left\| \sum_{\varphi \in \mathscr{A}} f_\varphi(x) \varphi(x) \right\|^2 d\mu(X) = \int_X \mathbf{1}_{E_{m,n}}(x) \left\| \sum_{\varphi \in \mathscr{A}} d_{m,\varphi} \varphi(x) \right\|^2 d\mu(X)$$

$$\geq \int_X \mathbf{1}_{E_{m,n}}(x) \cdot \left(B + \frac{1}{n}\right) \sum_{\varphi \in \mathscr{A}} |d_{m,\varphi}|^2 \, d\mu(x) = \left(B + \frac{1}{n}\right) \int_X \sum_{\varphi \in \mathscr{A}} |d_{m,\varphi} \mathbf{1}_{E_{m,n}}(x)|^2 \, d\mu(x)$$

$$= \left(B + \frac{1}{n}\right) \sum_{\varphi \in \mathscr{A}} \int_X |f_\varphi(x)|^2 \, d\mu(x).$$

Thus the upper bound in (iii) fails. In other words, the upper bound in (iii) implies the upper bound in (v). A similar argument applies for the lower bounds.

(v) $\implies$ (ii). Let $(g_i)_{i \in I}$ be a Riesz sequence in $L^2(X)$ with bounds a and b, and let $(c_{i,\varphi})_{i \in I, \varphi \in \mathscr{A}}$ be a sequence in $l^2(I \times \mathscr{A})$ having only finitely many nonzero terms. If (v) holds, then for a.e. $x \in X$ we apply the Riesz condition with the sequence $(\sum_{i \in I} c_{i,\varphi} g_i(x))_{\varphi \in \mathscr{A}}$ to deduce

$$A \sum_{\varphi \in \mathscr{A}} \left| \sum_{i \in I} c_{i,\varphi} g_i(x) \right|^2 \leq \left\| \sum_{\varphi \in \mathscr{A}} \left(\sum_{i \in I} c_{i,\varphi} g_i(x) \right) \varphi(x) \right\|_{\mathcal{H}}^2 \leq B \sum_{\varphi \in \mathscr{A}} \left| \sum_{i \in I} c_{i,\varphi} g_i(x) \right|^2.$$

Integrating this inequality over X produces

$$A \sum_{\varphi \in \mathscr{A}} \left\| \sum_{i \in I} c_{i,\varphi} g_i \right\|_{L^2(X)}^2 \leq \left\| \sum_{\varphi \in \mathscr{A}} \sum_{i \in I} c_{i,\varphi} g_i \varphi \right\|_{L^2(X;\mathcal{H})}^2 \leq B \sum_{\varphi \in \mathscr{A}} \left\| \sum_{i \in I} c_{i,\varphi} g_i \right\|_{L^2(X)}^2. \tag{2.5}$$

Meanwhile, for any $\varphi \in \mathscr{A}$ we apply the Riesz condition in $L^2(X)$ to deduce

$$a \sum_{i \in I} |c_{i,\varphi}|^2 \leq \left\| \sum_{i \in I} c_{i,\varphi} g_i \right\|_{L^2(X)}^2 \leq b \sum_{i \in I} |c_{i,\varphi}|^2.$$

Adding over all $\varphi \in \mathscr{A}$ and combining with (2.5) gives

$$aA \sum_{\varphi \in \mathscr{A}} \sum_{i \in I} |c_{i,\varphi}|^2 \leq \left\| \sum_{\varphi \in \mathscr{A}} \sum_{i \in I} c_{i,\varphi} g_i \varphi \right\|_{L^2(X;\mathcal{H})}^2 \leq bB \sum_{\varphi \in \mathscr{A}} \sum_{i \in I} |c_{i,\varphi}|^2.$$

In other words, $\{g_i \varphi \colon i \in I \text{ and } \varphi \in \mathscr{A}\}$ is a Riesz sequence with bounds aA and bB.

(ii) $\implies$ (i). This is immediate.

(i) $\implies$ (iii). Suppose $L^2(X)$ has an orthonormal basis $\mathcal{D}$ for which

$$\{g\varphi\colon g \in \mathcal{D}, \varphi \in \mathscr{A}\}$$

is a Riesz sequence with bounds A, B. First, let $(p_\varphi)_{\varphi\in\mathscr{A}}$ be a family in the finite linear span of $\mathcal{D} \subseteq L^2(X)$, with $p_\varphi \neq 0$ for only finitely many φ. Writing

$$p_\varphi(x) = \sum_{g\in\mathcal{D}} c_{g,\varphi}\, g(x)$$

for appropriate constants $c_{g,\varphi}$, we have (by definition)

$$\int_X \left\| \sum_{\varphi\in\mathscr{A}} p_\varphi(x)\varphi(x) \right\|^2 d\mu(x) = \left\| \sum_{\varphi\in\mathscr{A}} \sum_{g\in\mathcal{D}} c_{g,\varphi}\, g\varphi \right\|^2$$

and (by Parseval's identity)

$$\sum_{\varphi\in\mathscr{A}} \int_X |p_\varphi(x)|^2\, d\mu(x) = \sum_{\varphi\in\mathscr{A}} \sum_{g\in\mathcal{D}} |c_{g,\varphi}|^2.$$

Since $\{g\varphi\colon g \in \mathcal{D}, \varphi \in \mathscr{A}\}$ is a Riesz sequence, and since $c_{g,\varphi} \neq 0$ for only finitely many indices (g, φ),

$$A \sum_{\varphi\in\mathscr{A}} \int_X |p_\varphi(x)|^2\, d\mu(x) \leq \int_X \left\| \sum_{\varphi\in\mathscr{A}} p_\varphi(x)\varphi(x) \right\|^2 d\mu(x) \leq B \sum_{\varphi\in\mathscr{A}} \int_X |p_\varphi(x)|^2\, d\mu(x), \tag{2.6}$$

Now let $(f_\varphi)_{\varphi\in\mathscr{A}}$ be a family of functions in $L^2(X)$, as in (iii). For each $\varphi \in \mathscr{A}$, there is a sequence $\{p_{\varphi,k}\}_{k=1}^\infty$ of functions in the finite linear span of $\mathcal{D}$ such that $p_{\varphi,k} \to f_\varphi$ in $L^2(X)$. By passing to a subsequence if necessary, we may assume that $p_{\varphi,k}(x) \to f_\varphi(x)$ almost everywhere on X. Moreover, we can assume that $p_{\varphi,k} = 0$

when $f_\varphi = 0$, so that for each k, only finitely many $p_{\varphi,k} \neq 0$. By Fatou's Lemma and (2.6),

$$\int_X \left\| \sum_{\varphi \in \mathscr{A}} f_\varphi(x)\varphi(x) \right\|^2 d\mu(x) \leq \liminf_{k\to\infty} \int_X \left\| \sum_{\varphi \in \mathscr{A}} p_{\varphi,k}(x)\varphi(x) \right\|^2 d\mu(x)$$

$$\leq \liminf_{k\to\infty} B \sum_{\varphi \in \mathscr{A}} \int_X |p_{\varphi,k}(x)|^2 \, d\mu(x) = B \sum_{\varphi \in \mathscr{A}} \int_X |f_\varphi(x)|^2 \, d\mu(x).$$

In other words, the upper bound holds in (2.4).

It remains to prove the lower bound. To do this, we will upgrade the first inequality above to an equality. Previously, we showed that the upper bound in (iii) implies the upper bound in (v). Thus, for any sequence $(c_\varphi)_{\varphi\in\mathscr{A}}$ in $l^2(\mathscr{A})$ having only finitely many nonzero entries,

$$\left\| \sum_{\varphi \in \mathscr{A}} c_\varphi \varphi(x) \right\|^2 \leq B \sum_{\varphi \in \mathscr{A}} |c_\varphi|^2 \quad \text{for a.e. } x \in X.$$

In particular, $\|\varphi(x)\|^2 \leq B$ for all $\varphi \in \mathscr{A}$ and a.e. $x \in X$. Therefore,

$$\int_X \|f_\varphi(x)\varphi(x) - p_{\varphi,k}(x)\varphi(x)\|^2 \, d\mu(x) = \int_X |f_\varphi(x) - p_{\varphi,k}(x)|^2 \, \|\varphi(x)\|^2 \, d\mu(x)$$

$$\leq B \int_X |f_\varphi(x) - p_{\varphi,k}(x)|^2 \, d\mu(x).$$

Since $p_{\varphi,k} \to f_\varphi$ in $L^2(X)$, $p_{\varphi,k}\varphi \to f_\varphi\varphi$ in $L^2(X;\mathcal{H})$. In particular,

$$\int_X \|p_{\varphi,k}(x)\varphi(x)\|^2 \, d\mu(x) \to \int_X \|f_\varphi(x)\varphi(x)\|^2 \, d\mu(x).$$

Now (2.4) follows from (2.6).

(iii) $\iff$ (iv). Obviously (iv) implies (iii). Suppose conversely that (iii) holds. Without loss of generality, we may assume that $\mathscr{A}$ is infinite, and then we can enumerate $\mathscr{A} = (\varphi_k)_{k=1}^{\infty}$. Let $(f_k)_{k=1}^{\infty}$ be a sequence of functions in $L^2(X)$ such that $\sum_{k=1}^{\infty} \|f_k\|_{L^2(X)}^2 < \infty$. By Tonelli's Theorem,

$$\int_X \sum_{k=1}^{\infty} |f_k(x)|^2 \, d\mu(x) = \sum_{k=1}^{\infty} \int_X |f_k(x)|^2 \, d\mu(x) < \infty,$$

so $(f_k(x))_{k=1}^{\infty} \in l^2(\mathbb{N})$ for a.e. $x \in X$. We have shown that (iii) implies (v). Thus $(\varphi_k(x))_{k=1}^{\infty}$ is a Riesz sequence for a.e. $x \in X$. Applying the synthesis operator, we find that the sum $\sum_{k=1}^{\infty} f_k(x)\varphi_k(x)$ converges unconditionally for a.e. $x \in X$.

For each $n \in \mathbb{N}$, (iii) gives

$$A \sum_{k=1}^{n} \int_X |f_k(x)|^2 \, d\mu(x) \le \int_X \left\| \sum_{k=1}^{n} f_k(x)\varphi_k(x) \right\|^2 d\mu(x) \le B \sum_{k=1}^{n} \int_X |f_k(x)|^2. \tag{2.7}$$

Moreover, Fatou's Lemma and another application of (iii) show that

$$\int_X \left\| \sum_{k=1}^{\infty} f_k(x)\varphi_k(x) - \sum_{k=1}^{n} f_k(x)\varphi_k(x) \right\|^2 d\mu(x) = \int_X \lim_{N\to\infty} \left\| \sum_{k=n+1}^{N} f_k(x)\varphi_k(x) \right\|^2 d\mu(x)$$

$$\le \liminf_{N\to\infty} \int_X \left\| \sum_{k=n+1}^{N} f_k(x)\varphi_k(x) \right\|^2 d\mu(x) \le \liminf_{N\to\infty} B \sum_{k=n+1}^{N} \int_X |f_k(x)|^2 \, d\mu(x)$$

$$= B \sum_{k=n+1}^{\infty} \|f_k\|_{L^2(X)}^2 .$$

Thus $\sum_{k=1}^{n} f_k\varphi_k \to \sum_{k=1}^{\infty} f_k\varphi_k$ in $L^2(X;\mathcal{H})$-norm. Completeness shows that $\sum_{k=1}^{\infty} f_k\varphi_k \in L^2(X;\mathcal{H})$, and continuity of the norm gives

$$\lim_{n\to\infty}\int_X \left\|\sum_{k=1}^{n} f_k(x)\varphi_k(x)\right\|^2 d\mu(x) = \int_X \left\|\sum_{k=1}^{\infty} f_k(x)\varphi_k(x)\right\|^2 d\mu(x).$$

Sending $n\to\infty$ in (2.7) establishes (iv). $\square$

Remark 2.1.4. The theorem above holds when $\mu(X)=\infty$, but only vacuously. Indeed, the hypothesis that $\mu(X)<\infty$ was never used. However, if $\mathscr{A}$ is any countable family in $L^2(X;\mathcal{H})$ satisfying (v), then $\|\varphi(x)\|^2 \geq A$ for every $\varphi\in\mathscr{A}$ and a.e. $x\in X$. Hence,

$$\int_X \|\varphi(x)\|^2\, d\mu(x) \geq A\,\mu(X) = \infty$$

for each $\varphi\in\mathscr{A}$, so that $\mathscr{A}=\emptyset$.

2.12. Continuous frames

Definition 2.1.5. Let $\mathcal{H}$ be a Hilbert space, and let $(\mathcal{M},\mu_{\mathcal{M}})$ be a measure space. A family of vectors $(u_t)_{t\in\mathcal{M}}$ in $\mathcal{H}$ is called a *continuous frame* over $\mathcal{M}$ for $\mathcal{H}$ if both of the following hold:

(i) For each $v\in\mathcal{H}$, the function $t\mapsto\langle v,u_t\rangle$ is measurable $\mathcal{M}\to\mathbb{C}$.

(ii) There are constants $0<A\leq B<\infty$, called *frame bounds*, such that for each $v\in\mathcal{H}$,

$$A\|v\|^2 \leq \int_{\mathcal{M}} |\langle v,u_t\rangle|^2\, d\mu_{\mathcal{M}}(t) \leq B\|v\|^2.$$

When $A=B$, the frame is called *tight*, and when $A=B=1$, it is a *continuous Parseval frame*.

In practice, it is enough to check condition (ii) for v in a dense subset of $\mathcal{H}$. See [53, Proposition 2.5]. Continuous frames were introduced independently by Kaiser [48] and Ali, Antoine, and Gazou [2]. When $\mathcal{M}$ is a countable set and $\mu_{\mathcal{M}}$ is counting measure, continuous frames reduce to the usual discrete version.

Definition 2.1.6. Let (X, μ_X) be a measure space. A *Parseval determining set* for $L^1(X)$ consists of another measure space $(\mathcal{M}, \mu_{\mathcal{M}})$ and a family of functions $(g_t)_{t\in\mathcal{M}} \subseteq L^\infty(X)$ such that for each $f \in L^1(X)$, the mapping

$$t \mapsto \int_X f(x)\overline{g_t(x)}\, d\mu_X(x)$$

is measurable on $\mathcal{M}$, and

$$\int_{\mathcal{M}} \left| \int_X f(x)\overline{g_t(x)}\, d\mu_X(x) \right|^2 d\mu_{\mathcal{M}}(t) = \int_X |f(x)|^2\, d\mu_X(x). \tag{2.8}$$

We allow that both sides may be infinite.

This definition axiomatizes Lemma 3.5 of [13]. If $(g_t)_{t\in\mathcal{M}}$ is a Parseval determining set for $L^1(X)$, then so is $(\overline{g_t})_{t\in\mathcal{M}}$, by taking complex conjugates in the integrands above. It follows easily that a Parseval determining set is a determining set in the sense of Definition 2.1.1.

In the sections that follow, our primary example of a Parseval determining set will be the characters of an LCA group; see Lemma 3.2.2 infra. For another example, suppose that X is equipped with counting measure. If $\mathcal{M}$ is a countable set with counting measure, then a family $(g_t)_{t\in\mathcal{M}}$ in $l^2(X) \subseteq l^\infty(X)$ is a Parseval determining set for $L^1(X)$ if and only if

$$\sum_{t\in\mathcal{M}}\left|\sum_{x\in X} f(x)\overline{g_t(x)}\right|^2 = \sum_{x\in X}|f(x)|^2 \quad \text{for all } f\in l^1(X)\subseteq l^2(X),$$

if and only if $(g_t)_{t\in\mathcal{M}}$ is a discrete Parseval frame for $l^2(X)$.

If we relax our conditions and allow (X,μ) to be an arbitrary σ-finite measure space, then any Parseval determining set $(g_t)_{t\in\mathcal{M}}$ in $L^\infty(X)\cap L^2(X)$ satisfies the Parseval condition on the dense subspace $L^1(X)\cap L^2(X)\subseteq L^2(X)$, so it is a Parseval frame for $L^2(X)$. However, not every Parseval frame for $L^2(X)$ consisting of functions in $L^\infty(X)$ is a Parseval determining set for $L^1(X)$.[1] Indeed, for any $f\in L^1([0,1])\setminus L^2([0,1])$ there is an orthonormal basis $(g_n)_{n=1}^\infty\subseteq C([0,1])$ for $L^2([0,1])$ such that $\int f\overline{g_n}=0$ for every n; see [47, Satz 613]. For these functions, the left hand side of (2.8) is infinite, but the right hand side is zero.

Definition 2.1.7. Let (X,μ_X) and $(\mathcal{M},\mu_{\mathcal{M}})$ be measure spaces, and let $\mathcal{H}$ be a separable Hilbert space. We say a family $(\varphi_t)_{t\in\mathcal{M}}$ in $L^2(X;\mathcal{H})$ is *jointly measurable* if there is a function $\Phi\colon\mathcal{M}\times X\to\mathcal{H}$ satisfying the conditions:

(i) For a.e. $t\in\mathcal{M}$, $\Phi(t,\cdot)=\varphi_t$ a.e. on X.

(ii) For any $u\in\mathcal{H}$, the function $(t,x)\mapsto\langle\Phi(t,x),u\rangle$ is measurable on $\mathcal{M}\times X$.

By Pettis's Measurability Theorem [51, Theorem 1.1], condition (ii) says precisely that $\Phi\colon\mathcal{M}\times X\to\mathcal{H}$ is measurable with respect to the Borel σ-algebra on $\mathcal{H}$. In the case where $\mathcal{H}=L^2(Y)$, this is equivalent to another kind of pointwise measurability property, which we describe in Corollary 2.1.9 below.

[1] The author thanks Prof. Alexander Olevskii for his help answering this question.

Intuitively, joint measurability means that the function $(t, x) \mapsto \varphi_t(x)$ is measurable on $\mathcal{M} \times X$. However, this notion may depend on the choice of representative functions $\varphi_t \colon X \to \mathcal{H}$. In the sequel, we will often ignore this subtlety and integrate expressions involving $\varphi_t(x)$ over $\mathcal{M} \times X$. When this happens, it is to be assumed that we have fixed a measurable function $\Phi \colon \mathcal{M} \times X \to \mathcal{H}$ as above.

We expect that the next proposition is already known. However, we have not been able to locate a reference. Therefore, we supply a proof.

Proposition 2.1.8. *Let $(\mathcal{M}, \mu_{\mathcal{M}})$ and (Y, μ_Y) be σ-finite measure spaces, with $(\mathcal{M}, \mu_{\mathcal{M}})$ complete. For a family $(f_t)_{t \in \mathcal{M}}$ in $L^2(Y)$, the following are equivalent:*

(i) There is a measurable function $F \colon \mathcal{M} \times Y \to \mathbb{C}$ such that, for a.e. $t \in \mathcal{M}$, $F(t, \cdot) = f_t$ a.e. on Y.

(ii) For each $g \in L^2(Y)$, the function $t \mapsto \langle f_t, g \rangle$ is measurable on $\mathcal{M}$.

Proof. First assume that (i) holds. Find a sequence of simple measurable functions $S_n \colon \mathcal{M} \times Y \to \mathbb{C}$ such that $S_n(t, y) \to F(t, y)$ for all $(t, y) \in \mathcal{M}$, with $|S_n(t, y)| \leq |F(t, y)|$. Using the σ-finite conditions, we may assume that each S_n has support contained in a measurable rectangle with finite measure. For every $g \in L^2(Y)$ and every n, Hölder's Inequality yields

$$\begin{aligned}
&\int_{\mathcal{M}} \int_Y |S_n(t, y)\overline{g(y)}| \, d\mu_Y(y) \, d\mu_{\mathcal{M}}(t) \\
&\qquad\qquad \leq \int_{\mathcal{M}} \left(\int_Y |S_n(t, y)|^2 \, d\mu_Y(y) \right)^{1/2} \left(\int_Y |g(y)|^2 \, d\mu_Y(y) \right)^{1/2} d\mu_{\mathcal{M}}(t) \\
&\qquad\qquad < \infty.
\end{aligned}$$

Therefore Fubini's Theorem applies to the function $(t, y) \mapsto S_n(t, y)\overline{g(y)}$, and in particular the function

$$t \mapsto \int_Y S_n(t, y)\overline{g(y)}\, d\mu_Y(y)$$

is well defined a.e. and measurable on $\mathcal{M}$. Now the Lebesgue Dominated Convergence Theorem shows that

$$\langle f_t, g\rangle = \int_Y F(t, y)\overline{g(y)}\, d\mu_Y(y) = \lim_{n\to\infty} \int_Y S_n(t, y)\overline{g(y)}\, d\mu_Y(y)$$

for a.e. $t \in \mathcal{M}$. Hence the function $t \mapsto \langle f_t, g\rangle$ is the a.e. pointwise limit of measurable functions, and is itself measurable.

Suppose conversely that (ii) holds. By Pettis's Measurability Theorem, the function $t \mapsto f_t$ is measurable $\mathcal{M} \to L^2(Y)$; hence $t \mapsto \|f_t\|$ is measurable on $\mathcal{M}$. An easy exercise now shows that the measurable space $\mathcal{M}$ admits another measure $\tilde{\mu}_{\mathcal{M}}$ for which

$$\int_{\mathcal{M}} \|f_t\|^2 \, d\tilde{\mu}_{\mathcal{M}}(t) < \infty.$$

Since we are concerned only with measurability, we may replace $\mu_{\mathcal{M}}$ with $\tilde{\mu}_{\mathcal{M}}$ and assume that the function $t \mapsto f_t$ belongs to $L^2(\mathcal{M}; L^2(Y))$. The usual identification of $L^2(\mathcal{M} \times Y)$ with $L^2(\mathcal{M}; L^2(Y))$ now proves (i). (See for instance [54, Theorem II.10(c)].) $\square$

Corollary 2.1.9. *Let (X, μ_X), (Y, μ_Y), and $(\mathcal{M}, \mu_{\mathcal{M}})$ be complete, σ-finite measure spaces. A family $(\varphi_t)_{t\in\mathcal{M}}$ in $L^2(X; L^2(Y))$ is jointly measurable if and only if there is a measurable function $\Phi\colon \mathcal{M} \times X \times Y \to \mathbb{C}$ such that for a.e. $t \in \mathcal{M}$, for a.e. $x \in X$, $\Phi(t, x, \cdot) = \varphi_t(x)$ a.e. on Y. Consequently, the notion of "joint measurability" remains* *unchanged when we identify $L^2(X; L^2(Y))$ with $L^2(X \times Y) = L^2(X \times Y; \mathbb{C})$, or with $L^2(Y; L^2(X))$.*

Proof. Apply Proposition 2.1.8 to the family $(\varphi_t(x))_{(t,x)\in\mathcal{M}\times X} \subseteq L^2(Y)$. We leave it to the reader to check the details surrounding sets of measure zero. $\square$

The next theorem is an abstract version of [11, Theorem 2.3(i)], whose argument we follow. See also [14, Theorem 4.2] and [13, Theorem 5.1].

Theorem 2.1.10. *Let (X, μ_X) and $(\mathcal{M}, \mu_{\mathcal{M}})$ be σ-finite measure spaces, and let $\mathcal{D} = (g_s)_{s\in\mathcal{M}}$ be a Parseval determining set for $L^1(X)$. Fix a separable Hilbert space $\mathcal{H}$, another σ-finite measure space $(\mathcal{N}, \mu_{\mathcal{N}})$, and a jointly measurable familiy $\mathscr{A} = (\varphi_t)_{t\in\mathcal{N}}$ in $L^2(X;\mathcal{H})$. Let $\mathscr{A}_0 \subseteq \mathscr{A}$ be a countable dense subset, and define J as in (2.1). For constants $0 < A \le B < \infty$, the following are equivalent:*

(i) $E_{\mathcal{D}}(\mathscr{A})$ forms a continuous frame for $S_{\mathcal{D}}(\mathscr{A})$ over $\mathcal{M}\times\mathcal{N}$, with bounds A and B. That is,

$$A\int_X \|\psi(x)\|^2\, d\mu_X(x) \le \int_{\mathcal{N}}\int_{\mathcal{M}} \left|\int_X \langle \psi(x), g_s(x)\varphi_t(x)\rangle\, d\mu_X(x)\right|^2 d\mu_{\mathcal{M}}(s)\, d\mu_{\mathcal{N}}(t)$$

$$\le B\int_X \|\psi(x)\|^2\, d\mu_X(x)$$

for all $\psi \in S_{\mathcal{D}}(\mathscr{A})$.

(ii) For a.e. $x \in X$ and every $u \in J(x)$,

$$A\|u\|^2 \le \int_{\mathcal{N}} |\langle u, \varphi_t(x)\rangle|^2 d\mu_{\mathcal{N}}(t) \le B\|u\|^2 .$$

We are tempted to interpret condition (ii) to mean that the family $((\varphi_t)(x))_{t\in\mathcal{N}}$ forms a continuous frame for $J(x)$ for a.e. $x \in X$. However, when $\mathscr{A}$ is uncountable, the vectors $\varphi_t(x)$ need not reside in $J(x)$. A more precise interpretation says that $\{P_J(x)[\varphi_t(x)] \colon t \in \mathcal{N}\}$ forms a continuous frame for $J(x)$ for a.e. $x \in X$.

The theorem significantly reduces the problem of determining when $E_{\mathcal{D}}(\mathscr{A})$ forms a continuous frame. For instance, when $\mathscr{A} \subseteq L^2(X;\mathcal{H})$ is a countable family equipped with counting measure, condition (ii) says that for a.e. $x \in X$, $\{\varphi(x) \colon \varphi \in \mathscr{A}\}$ forms a discrete frame for $J(x)$. Thus, a continuous problem in $L^2(X;\mathcal{H})$ reduces to a discrete problem in $\mathcal{H}$. The reduction is even more pronounced when $\mathscr{A}$ consists of a single function $\varphi \in L^2(X;\mathcal{H})$. In that case, (ii) reduces to

(ii') For a.e. $x \in X$, either $\varphi(x) = 0$ or $A \leq \|\varphi(x)\|^2 \leq B$.

Proof of Theorem 2.1.10. Joint measurability of $\mathscr{A}$ ensures that the integrals above are well defined; use Tonelli's Theorem for the integral in condition (ii). For each $\psi \in S_{\mathcal{D}}(\mathscr{A})$, we compute

$$\begin{aligned}
&\int_{\mathcal{N}} \int_{\mathcal{M}} \left| \int_X \langle \psi(x), g_s(x)\varphi_t(x) \rangle \, d\mu_X(x) \right|^2 d\mu_{\mathcal{M}}(s) \, d\mu_{\mathcal{N}}(t) \qquad (2.9)\\
&\qquad = \int_{\mathcal{N}} \int_{\mathcal{M}} \left| \int_X \langle \psi(x), \varphi_t(x) \rangle \overline{g_s(x)} \, d\mu_X(x) \right|^2 d\mu_{\mathcal{M}}(s) d\mu_{\mathcal{N}}(t)\\
&\qquad = \int_{\mathcal{N}} \int_X |\langle \psi(x), \varphi_t(x) \rangle|^2 \, d\mu_X(x) \, d\mu_{\mathcal{N}}(t)\\
&\qquad = \int_X \int_{\mathcal{N}} |\langle \psi(x), \varphi_t(x) \rangle|^2 \, d\mu_{\mathcal{N}}(t) \, d\mu_X(x),
\end{aligned}$$

since $\mathcal{D}$ is a Parseval determining set for $L^1(X)$.

If (ii) holds, then (2.2) shows that

$$A \|\psi(x)\|^2 \leq \int_{\mathcal{N}} |\langle \psi(x), \varphi_t(x) \rangle|^2 \, d\mu_{\mathcal{N}}(t) \leq B \|\psi(x)\|^2$$

for all $\psi \in S_{\mathcal{D}}(\mathscr{A})$ and a.e. $x \in X$. Integrating over X and applying (2.9) proves (i).

Suppose conversely that (ii) fails. Fix a countable dense subset $\{u_m\}_{m=1}^{\infty} \subseteq \mathcal{H}$. For a.e. $x \in X$, it follows that $\{P_J(x)u_m\}_{m=1}^{\infty}$ is a dense subset of $J(x)$. Given $m, n \in \mathbb{N}$, define

$$E_{m,n} = \left\{ x \in X \colon \int_{\mathcal{N}} |\langle P_J(x)u_m, \varphi_t(x) \rangle|^2 \, d\mu_{\mathcal{N}}(t) > \left(B + \frac{1}{n} \right) \|P_J(x)u_m\|^2 \right\}$$
$$F_{m,n} = \left\{ x \in X \colon \int_{\mathcal{N}} |\langle P_J(x)u_m, \varphi_t(x) \rangle|^2 \, d\mu_{\mathcal{N}}(t) < \left(A - \frac{1}{n} \right) \|P_J(x)u_m\|^2 \right\},$$

each of which is well-defined up to a set of measure zero. For a.e. $x \notin \bigcup_{m,n=1}^{\infty} (E_{m,n} \cup F_{m,n})$,

$$A \|P_J(x)u_m\|^2 \leq \int_{\mathcal{N}} |\langle P_J(x)u_m, \varphi_t(x) \rangle|^2 \, d\mu_{\mathcal{N}}(t) \leq B \|P_J(x)u_m\|^2 \quad \text{for all } m \in \mathbb{N},$$

so that $\{P_J(x)[\varphi_t(x)]\}_{t \in \mathcal{N}}$ forms a frame for $J(x)$ with bounds A, B. Therefore at least one set $E_{m,n}$ or $F_{m,n}$ has positive measure. In the first case, fix a Borel set $E \subseteq E_{m,n}$ with $0 < \mu_X(E) < \infty$, and define $\theta \in L^2(X; \mathcal{H})$ by

$$\theta(x) = \mathbf{1}_E(x) \cdot P_J(x)u_m.$$

Since we used strict inequality in the definition of $E_{m,n}$, $\|\theta(x)\| > 0$ on E. Moreover, $\theta \in S_{\mathcal{D}}(\mathscr{A})$ by (2.2), and (2.9) shows that

$$\begin{aligned}
\int_{\mathcal{N}} \int_{\mathcal{M}} \left| \int_X \langle \theta(x), g_s(x)\varphi_t(x)\rangle \, d\mu_X(x) \right|^2 d\mu_{\mathcal{M}}(s) \, d\mu_{\mathcal{N}}(t) \\
&= \int_X \int_{\mathcal{N}} |\langle \theta(x), \varphi_t(x)\rangle|^2 \, d\mu_{\mathcal{N}}(t) \, d\mu_X(x) \\
&= \int_X \mathbf{1}_E(x) \cdot \int_{\mathcal{N}} |\langle P_J(x)u_m, \varphi_t(x)\rangle|^2 \, d\mu_{\mathcal{N}}(t) \, d\mu_X(x) \\
&\geq \int_X \mathbf{1}_E(x) \cdot \left(B + \frac{1}{n}\right) \|P_J(x)u_m\|^2 \, d\mu_X(x) \\
&= \left(B + \frac{1}{n}\right) \int_X \|\theta(x)\|^2 \, d\mu_X(x).
\end{aligned}$$

Thus (i) fails. A similar argument shows that (i) fails when $\mu_X(F_{m,n}) > 0$. This completes the proof. $\square$

2.2. A Weil formula for right cosets

Let G be a second countable locally compact group, and let $\Gamma \subseteq G$ be a closed subgroup. We emphasize that these groups need not be abelian. Our purpose is to examine the measure-theoretic interplay between G, Γ, and the topological quotients G/Γ and $\Gamma\backslash G$; the latter is the space of *right* cosets of Γ in G. Our main result is the existence of a measure on $\Gamma\backslash G$ for which $G \cong \Gamma \times \Gamma\backslash G$ as measure spaces, and for which the resulting unitary $U\colon L^2(G) \to L^2(\Gamma \times \Gamma\backslash G)$ is well behaved under left translation by Γ.

There is a positive regular Borel measure μ_G on G, called *(left) Haar measure*, such that

$$\int_G f(yx) \, d\mu_G(x) = \int_G f(x) \, d\mu_G(x) \tag{2.10}$$

for all $f \in L^1(G, \mu_G)$ and all $y \in G$. This measure is unique up to multiplication by a scalar $c > 0$. Fix a scale once and for all. Equation (2.10) generally fails if we replace yx with xy. However, there is a continuous function $\Delta_G \colon G \to (0, \infty)$, called the *modular function*, such that

$$\int_G f(xy)\, d\mu_G(x) = \Delta_G(y^{-1}) \int_G f(x)\, d\mu_G(x) \tag{2.11}$$

and

$$\int_G f(x^{-1})\, d\mu_G(x) = \int_G f(x)\Delta_G(x^{-1})\, d\mu_G(x) \tag{2.12}$$

for all $f \in L^1(G, \mu_G)$ and $y \in G$. When $\Delta_G \equiv 1$, G is called *unimodular*. The modular function is a homomorphism with respect to multiplication on $(0, \infty)$, and it is independent of the choice of Haar measure on G. The subgroup Γ also has a modular function Δ_Γ and a left Haar measure μ_Γ, whose scale we also fix.

A *rho function* for the pair (G, Γ) is a continuous map $\rho \colon G \to (0, \infty)$ with the property that

$$\rho(x\xi) = \rho(x) \frac{\Delta_\Gamma(\xi)}{\Delta_G(\xi)}$$

for all $x \in G$ and all $\xi \in \Gamma$. Such a function always exists; fix a choice once and for all, taking $\rho = 1$ if possible. There is a unique positive regular Borel measure $\mu_{G/\Gamma}$ on G/Γ such that

$$\int_G f(x)\rho(x)\, d\mu_G(x) = \int_{G/\Gamma} \int_\Gamma f(x\xi)\, d\mu_\Gamma(\xi)\, d\mu_{G/\Gamma}(x\Gamma) \tag{2.13}$$

for all $f \in L^1(G)$. See Folland [26, Section 2.6] and Reiter and Stegeman [55, Section 8.2]. In particular, the inner integral does not depend on the choice of coset representative for $x\Gamma$, the mapping $\xi \mapsto f(x\xi)$ belongs to $L^1(\Gamma)$ for $\mu_{G/\Gamma}$-a.e.

$x\Gamma \in G/\Gamma$, and the function $x\Gamma \mapsto \int_\Gamma f(x\xi)\, d\mu_\Gamma(\xi)$ is measurable on G/Γ. The associated measure $\mu_{G/\Gamma}$ is strongly quasi-invariant under the action of G, in the sense that

$$\int_{G/\Gamma} f(y^{-1}x\Gamma)\, d\mu_{G/\Gamma}(x\Gamma) = \int_{G/\Gamma} f(x\Gamma)\frac{\rho(yx)}{\rho(x)}\, d\mu_{G/\Gamma}(x\Gamma) \tag{2.14}$$

for all $f \in C_c(G/\Gamma)$ and all $y \in G$. In particular, $\mu_{G/\Gamma}$ is invariant under the action of G if and only if $\rho = 1$; this can happen if and only if $\Delta_\Gamma = \Delta_G|_\Gamma$. In that case, (2.13) becomes *Weil's formula*,

$$\int_G f(x)\, d\mu_G(x) = \int_{G/\Gamma}\int_\Gamma f(x\xi)\, d\mu_\Gamma(\xi)\, d\mu_{G/\Gamma}(x\Gamma) \quad \text{for all } f \in L^1(G). \tag{2.15}$$

For instance, when Γ is normal in G, any choice of left Haar measure on G/Γ is invariant under the action of G. Therefore we can take $\rho = 1$, and by (2.14), $\mu_{G/\Gamma}$ is the unique left Haar measure on G/Γ that satisfies (2.15).

By a well-known result of Feldman and Greenleaf [25], there is a Borel measurable function $\tau\colon G/\Gamma \to G$ with the property that $q \circ \tau = \mathrm{id}_{G/\Gamma}$, where q is the quotient mapping onto G/Γ. In effect, τ chooses a representative for each coset of Γ in G, and it does it in a measurable way. We call such a function a *Borel section* for G/Γ. To describe τ, it suffices to give its *fundamental domain* $\tau(G/\Gamma)$, since $\tau(x\Gamma)$ is the unique element of $\tau(G/\Gamma) \cap x\Gamma$. Moreover, τ is a Borel measurable function if and only if its fundamental domain is a Borel subset of G. As remarked in [25], τ can be chosen such that, whenever $K \subseteq G/\Gamma$ is compact, $\tau(K)$ has compact closure in G. Fix a section τ with this property once and for all, and let $T\colon \Gamma \times G/\Gamma \to G$ be the associated bijection

$$T(\xi, x\Gamma) = \tau(x\Gamma)\xi.$$

Proposition 2.2.1. *The function T described above is an isomorphism of measure spaces*

$$T\colon \left(\Gamma \times G/\Gamma,\, d\mu_\Gamma \otimes d\mu_{G/\Gamma}\right) \to (G,\, \rho\, d\mu_G)\,.$$

This result was stated in a section of notes by Folland [26, §2.7]. However, its proof was only sketched, and an important detail was missing. The full proof relies on the remarkable lemma below. We remind the reader that a separable topological space is called *Polish* if it admits a complete metric. Every second countable locally compact Hausdorff space is Polish, and the product of Polish spaces is Polish; see Kechris [48, Theorem 5.3, Proposition 3.3]. The lemma below is Theorem 14.12 of [48].

Lemma 2.2.2. *Let X and Y be Polish spaces. Then every Borel measurable bijection $f\colon X \to Y$ is an isomorphism of Borel spaces. That is, f^{-1} is also Borel measurable.*

Proof of Proposition 2.2.1. For each $\xi \in \Gamma$, the function $T_\xi\colon G/\Gamma \to G$ given by $T_\xi(x\Gamma) = T(\xi, x\Gamma) = \tau(x\Gamma)\xi$ is Borel measurable; and for each $x\Gamma \in G/\Gamma$, the function $T^{x\Gamma}\colon \Gamma \to G$ given by $T^{x\Gamma}(\xi) = T(\xi, x\Gamma) = \tau(x\Gamma)\xi$ is continuous. It follows that T is Borel measurable (see for instance [48, Exercise 11.3]). By Lemma 2.2.2 and (2.13), T is an isomorphism of measure spaces. $\square$

Corollary 2.2.3. *There is a unitary $U\colon L^2(G, \mu_G) \to L^2(\Gamma \times G/\Gamma, \mu_\Gamma \otimes \mu_{G/\Gamma})$ such that*

$$Uf(\xi, x\Gamma) = \frac{f(\tau(x\Gamma)\xi)}{\sqrt{\rho(\tau(x\Gamma)\xi)}}$$

for all $f \in L^2(G)$, μ_Γ-a.e. $\xi \in \Gamma$, and $\mu_{G/\Gamma}$-a.e. $x\Gamma \in G/\Gamma$.

Proof. Let $V\colon L^2(G,\, d\mu_G) \to L^2(G,\, \rho\, d\mu_G)$ be the isometric isomorphism given by $V(f) = f/\sqrt{\rho}$. Follow V by $W\colon L^2(G,\, \rho\, d\mu_G) \to L^2(\Gamma \times G/\Gamma,\, d\mu_\Gamma \otimes d\mu_{\Gamma\otimes G/\Gamma})$, $W(g) = g \circ T$. The resulting unitary is U. $\square$

If we are willing to sacrifice invariance under the action of G, we can eliminate the rho function in the results above by replacing left cosets with right. Denote $q_L \colon G \to G/\Gamma$ and $q_R \colon G \to \Gamma\backslash G$ for the respective quotient maps, and $\varepsilon \colon \Gamma\backslash G \to G/\Gamma$ for the homeomorphism $\varepsilon(\Gamma x) = x^{-1}\Gamma$. If $i \colon G \to G$ is the inversion map, one easily checks that $\varepsilon \circ q_R \circ i = q_L$, as shown below.

$$\begin{tikzcd}
G \arrow[r, "i"] \arrow[d, "q_L"] & G \arrow[d, "q_R"] \\
G/\Gamma \arrow[u, dotted, bend left, "\tau"] & \Gamma\backslash G \arrow[l, "\varepsilon"]
\end{tikzcd}$$

Define $\gamma \colon \Gamma\backslash G \to G$ by $\gamma = i \circ \tau \circ \varepsilon$, that is,

$$\gamma(\Gamma x) = \left[\tau(x^{-1}\Gamma)\right]^{-1}. \tag{2.16}$$

We claim that γ is a Borel section for $\Gamma\backslash G$. Indeed, $\varepsilon \circ q_R \circ i \circ \tau = q_L \circ \tau = id_{G/\Gamma}$, so $q_R \circ i \circ \tau = \varepsilon^{-1}$. Hence

$$q_R \circ \gamma = q_R \circ i \circ \tau \circ \varepsilon = \varepsilon^{-1} \circ \varepsilon = id_{\Gamma\backslash G}.$$

The claim follows once we observe that γ is formed by composing τ with homeomorphisms on either side, so it is Borel. Moreover, it inherits the property from τ, that whenever $K \subseteq \Gamma\backslash G$ is compact, $\gamma(K)$ has compact closure in G.

Theorem 2.2.4. *There is a unique positive regular Borel measure $\mu_{\Gamma\backslash G}$ on $\Gamma\backslash G$ such that*

$$\int_G f(x)\, d\mu_G(x) = \int_{\Gamma\backslash G} \int_\Gamma f\left(\xi\gamma(\Gamma x)\right)\, d\mu_\Gamma(\xi)\, d\mu_{\Gamma\backslash G}(\Gamma x) \tag{2.17}$$

for all $f \in L^1(G)$*. In particular,* $\xi \mapsto f(\xi\gamma(\Gamma x))$ *belongs to* $L^1(\Gamma)$ *for* $\mu_{\Gamma\backslash G}$*-a.e.* $\Gamma x \in \Gamma\backslash G$*, and the function* $\Gamma x \mapsto \int_\Gamma f(\xi\gamma(\Gamma x))\, d\mu_\Gamma(\xi)$ *is measurable on* $\Gamma\backslash G$*. If* $\varepsilon\colon \Gamma\backslash G \to G/\Gamma$ *is the homeomorphism* $\varepsilon(\Gamma x) = x^{-1}\Gamma$*, then*

$$\int_{\Gamma\backslash G} f(\Gamma x)\, d\mu_{\Gamma\backslash G}(\Gamma x) = \int_{G/\Gamma} f(\varepsilon^{-1}(x\Gamma))\, \frac{1}{\rho(\tau(x\Gamma))\Delta_G(\tau(x\Gamma))}\, d\mu_{G/\Gamma}(x\Gamma) \qquad (2.18)$$

for $f \in L^1(\Gamma\backslash G)$*.*

Proof. We'll first show that (2.18) defines a measure $\mu_{\Gamma\backslash G}$ on $\Gamma\backslash G$ satisfying (2.17). Recall that the image of a compact set in G/Γ has compact closure in G. If $f\colon G \to \mathbb{R}$ is any continuous function, it follows that $f \circ \tau\colon G/\Gamma \to \mathbb{R}$ is Borel measurable and bounded on compact subsets. In particular, $x\Gamma \mapsto \frac{1}{\rho(\tau(x\Gamma))\Delta_G(\tau(x\Gamma))}$ is a locally $\mu_{G/\Gamma}$-integrable function on G/Γ, and we can use it to define a positive regular Borel measure $d\tilde{\mu}_{G/\Gamma} = \frac{1}{(\rho\cdot\Delta_G)\circ\tau}\, d\mu_{G/\Gamma}$. Since $\varepsilon\colon \Gamma\backslash G \to G/\Gamma$ is a homeomorphism, there is a positive regular Borel measure $\mu_{\Gamma\backslash G}$ on $\Gamma\backslash G$ given by $d\mu_{\Gamma\backslash G}(\Gamma x) = d\tilde{\mu}_{G/\Gamma}(\varepsilon(\Gamma x))$. For a Borel set $E \subseteq \Gamma\backslash G$, this means that

$$\begin{aligned}\int_{\Gamma\backslash G} \mathbf{1}_E(\Gamma x)\, d\mu_{\Gamma\backslash G}(\Gamma x) &= \mu_{\Gamma\backslash G}(E) = \tilde{\mu}_{G/\Gamma}(\varepsilon(E))\\ &= \int_{G/\Gamma} \mathbf{1}_{\varepsilon(E)}(x\Gamma)\frac{1}{\rho(\tau(x\Gamma))\Delta_G(\tau(x\Gamma))}\, d\mu_{G/\Gamma}(x\Gamma)\\ &= \int_{G/\Gamma} \mathbf{1}_E(\varepsilon^{-1}(x\Gamma))\frac{1}{\rho(\tau(x\Gamma))\Delta_G(\tau(x\Gamma))}\, d\mu_{G/\Gamma}(x\Gamma).\end{aligned}$$

It follows that (2.18) holds for all $f \in L^1(\Gamma\backslash G, \mu_{\Gamma\backslash G})$.

Given $f \in L^1(G)$, use (2.13) to compute

$$\begin{aligned}
\int_G f(x)\,d\mu_G(x) &= \int_G \frac{f(x^{-1})}{\rho(x)} \Delta_G(x^{-1})\rho(x)\,d\mu_G(x) \\
&= \int_{G/\Gamma}\int_\Gamma \frac{f((x\xi)^{-1})}{\rho(x\xi)} \Delta_G((x\xi)^{-1})\,d\mu_\Gamma(\xi)\,d\mu_{G/\Gamma}(x\Gamma) \\
&= \int_{G/\Gamma}\int_\Gamma \frac{f(\xi^{-1}\tau(x\Gamma)^{-1})}{\rho(\tau(x\Gamma)\xi)} \Delta_G(\xi^{-1}\tau(x\Gamma)^{-1})\,d\mu_\Gamma(\xi)\,d\mu_{G/\Gamma}(x\Gamma) \\
&= \int_{G/\Gamma}\int_\Gamma \frac{f(\xi^{-1}\gamma(\Gamma x^{-1}))\Delta_G(\xi)}{\rho(\tau(x\Gamma))\Delta_\Gamma(\xi)} \Delta_G(\xi^{-1})\Delta_G(\tau(x\Gamma)^{-1})\,d\mu_\Gamma(\xi)\,d\mu_{G/\Gamma}(x\Gamma) \\
&= \int_{G/\Gamma}\int_\Gamma \frac{f\left(\xi^{-1}\gamma(\varepsilon^{-1}(x\Gamma))\right)}{\rho(\tau(x\Gamma))\Delta_G(\tau(x\Gamma))} \Delta_\Gamma(\xi^{-1})\,d\mu_\Gamma(\xi)\,d\mu_{G/\Gamma}(x\Gamma).
\end{aligned}$$

Using the inversion formula (2.12) on Γ, we obtain

$$\begin{aligned}
\int_G f(x)\,d\mu_G(x) &= \int_{G/\Gamma}\int_\Gamma f\left(\xi\gamma(\varepsilon^{-1}(x\Gamma))\right)\,d\mu_\Gamma(\xi)\,\frac{1}{\rho(\tau(x\Gamma))\Delta_G(\tau(x\Gamma))}\,d\mu_{G/\Gamma}(x\Gamma) \\
&= \int_{\Gamma\backslash G}\int_\Gamma f(\xi\gamma(\Gamma x))\,d\mu_\Gamma(\xi)\,d\mu_{\Gamma\backslash G}(\Gamma x).
\end{aligned}$$

Therefore $\mu_{\Gamma\backslash G}$ satisfies (2.17).

It remains to prove uniqueness. Recall from Folland [26, Proposition (2.48)] that the periodization operator $P\colon C_c(G) \to C_c(G/\Gamma)$ given by

$$(Pf)(x\Gamma) = \int_\Gamma f(\tau(x\Gamma)\xi)\,d\mu_\Gamma(\xi)$$

is surjective, and if $\phi \in C_c(G/\Gamma)$ is nonnegative, we can find nonnegative $f \in C_c(G)$ with $Pf = \phi$. By an argument analogous to the one given in [26, Proposition (2.48)], there is a surjective operator $\tilde{P}\colon C_c(G) \to C_c(\Gamma\backslash G)$ given by

$$(\tilde{P}f)(\Gamma x) = \int_\Gamma f(\xi\gamma(\Gamma x))\, d\mu_\Gamma(\xi),$$

and for $\phi \in C_c(\Gamma\backslash G)$ with $\phi \geq 0$, we can find $f \geq 0$ in $C_c(G)$ with $\tilde{P}f = \phi$; we leave it to the reader to make the necessary adjustments. Moreover, (2.17) shows that when $f, g \in C_c(G)$ are functions with $\tilde{P}f = \tilde{P}g$, $\int_G f(x)\, d\mu_G(x) = \int_G g(x)\, d\mu_G(x)$. Therefore $\tilde{P}f \mapsto \int_G f(x)\, d\mu_G(x)$ is a well-defined positive linear functional on $C_c(\Gamma\backslash G)$, and by the uniqueness in the Riesz Representation Theorem, there is only one positive regular Borel measure $\mu_{\Gamma\backslash G}$ satisfying (2.17). $\square$

A word of warning: this measure is *not* usually invariant under the right action of G, even when an invariant measure exists, unless G is unimodular. Indeed, a right invariant measure on $\Gamma\backslash G$, suitably normalized, would cause (2.17) to hold with *right* Haar measure on G in place of the *left* Haar measure μ_G. A staightforward (but tedious) computation involving (2.18) and (2.14) produces

$$\begin{aligned}&\int_{\Gamma\backslash G} f(\Gamma xy)\, d\mu_{\Gamma\backslash G}(\Gamma x)\\ &\qquad = \int_{\Gamma\backslash G} f(\Gamma x) \frac{\rho\left(\gamma(\Gamma x)^{-1}\right) \Delta_G\left(\gamma(\Gamma x)^{-1}\right)}{\rho\left(\gamma(\Gamma xy^{-1})^{-1}\right) \Delta_G\left(\gamma(\Gamma xy^{-1})^{-1}\right)} \frac{\rho\left(yx^{-1}\right)}{\rho\left(x^{-1}\right)}\, d\mu_{\Gamma\backslash G}(\Gamma x)\end{aligned}$$

for all $f \in C_c(\Gamma\backslash G)$ and $y \in G$.

Remark 2.2.5. When Γ is discrete and μ_Γ is counting measure, γ identifies $(\Gamma\backslash G, \mu_{\Gamma\backslash G})$ as a measure space with $(\gamma(\Gamma\backslash G), \mu_G)$, but when Γ is not discrete, $\mu_G(\gamma(\Gamma\backslash G)) = 0$. To see this, let $E \subseteq \gamma(\Gamma\backslash G)$ be a Borel set with $\mu_G(E) < \infty$, and use (2.17) to compute

$$\begin{aligned}\mu_G(E) &= \int_G \mathbf{1}_E(x)\,d\mu_G(x) = \int_{\Gamma\backslash G}\int_\Gamma \mathbf{1}_E(\xi\gamma(\Gamma x))\,d\mu_\Gamma(\xi)\,d\mu_{\Gamma\backslash G}(\Gamma x)\\ &= \int_{\Gamma\backslash G} \mu_\Gamma(\{1\})\cdot \mathbf{1}_E(\gamma(\Gamma x))\,d\mu_{\Gamma\backslash G}(\Gamma x) = \mu_\Gamma(\{1\})\int_{\Gamma\backslash G}\mathbf{1}_{\gamma^{-1}(E)}(\Gamma x)\,d\mu_{\Gamma\backslash G}(\Gamma x)\\ &= \mu_\Gamma(\{1\})\cdot\mu_{\Gamma\backslash G}(\gamma^{-1}(E)).\end{aligned}$$

Lemma 2.2.2 shows that γ preserves the Borel σ-algebra, and the claim follows.

Theorem 2.2.6. *The mapping*

$$T\colon \big(\Gamma\times\Gamma\backslash G,\, d\mu_\Gamma\otimes d\mu_{\Gamma\backslash G}\big)\to (G,\, d\mu_G)$$

given by $T(\xi,\Gamma x)=\xi\gamma(\Gamma x)$ is a measure space isomorphism.

Proof. It follows from Lemma 2.2.2 and (2.17) just as Proposition 2.2.1 did. □

Corollary 2.2.7. *There is a unitary map $U\colon L^2(G,\mu_G)\to L^2(\Gamma\times\Gamma\backslash G,\mu_\Gamma\otimes\mu_{\Gamma\backslash G})$ such that*

$$Uf(\xi,\Gamma x) = f(\xi\gamma(\Gamma x))$$

for all $f\in L^2(G)$, μ_Γ-a.e. $\xi\in\Gamma$, and $\mu_{\Gamma\backslash G}$-a.e. $\Gamma x\in\Gamma\backslash G$.

Remark 2.2.8. When Γ is a *normal* subgroup of G, we have defined two measures on the coinciding quotient spaces $G/\Gamma=\Gamma\backslash G$, namely $\mu_{G/\Gamma}$ and $\mu_{\Gamma\backslash G}$. In the most general setting, these measures need not be equal, but they are related in a way that we now describe. For arbitrary $f\in C_c(G/\Gamma)$, we compute

$$\int_{\Gamma\backslash G} f(\Gamma x)\,d\mu_{\Gamma\backslash G}(\Gamma x) = \int_{G/\Gamma} f(\Gamma x^{-1})\Delta_G(\tau(x\Gamma)^{-1})d\mu_{G/\Gamma}(x\Gamma)$$

$$= \int_{G/\Gamma} f(\Gamma x^{-1}) \Delta_G(\gamma(\Gamma x^{-1})) d\mu_{G/\Gamma}(x\Gamma).$$

Since Γ is normal in G, $\mu_{G/\Gamma}$ is a left Haar measure on G/Γ. Denoting $\Delta_{G/\Gamma}$ for the modular function on G/Γ, and identifying Γx^{-1} with $x^{-1}\Gamma$, we compute

$$\int_{\Gamma \backslash G} f(\Gamma x)\, d\mu_{\Gamma\backslash G}(\Gamma x) = \int_{G/\Gamma} f(x\Gamma) \Delta_G(\gamma(\Gamma x)) \Delta_{G/\Gamma}(x^{-1}\Gamma)\, d\mu_{G/\Gamma}(x\Gamma),$$

by (2.12). Thus,

$$d\mu_{\Gamma\backslash G} = \frac{\Delta_G \circ \gamma}{\Delta_{G/\Gamma}}\, d\mu_{G/\Gamma}. \tag{2.19}$$

There is another way to compute the Radon-Nikodym derivative that is sometimes useful. For each $x \in G$, there is a unique number $\delta(x) > 0$ such that

$$\int_\Gamma f(x\xi x^{-1})\, d\mu_\Gamma(\xi) = \delta(x) \int_\Gamma f(\xi)\, d\mu_\Gamma(\xi)$$

for all $f \in L^1(\Gamma)$. In fact, $\delta(x) = \frac{\Delta_G(x)}{\Delta_{G/\Gamma}(x\Gamma)}$. (See Nachbin [50, Chapter II, Propositions 16 and 22].) Thus we can take any function $f \in L^1(\Gamma)$ with nonzero integral that we like, and compute

$$\frac{\Delta_G(\gamma(\Gamma x))}{\Delta_{G/\Gamma}(x\Gamma)} = \delta(\gamma(x\Gamma)) = \int_\Gamma f\left(\gamma(\Gamma x)\xi\gamma(\Gamma x)^{-1}\right)\, d\mu_\Gamma(\xi) \cdot \left(\int_\Gamma f(\xi)\, d\mu_\Gamma(\xi)\right)^{-1}.$$

For instance, in the case where Γ is compact and normal, we can take $f = 1$ in the formula above to see that $\frac{\Delta_G \circ \gamma}{\Delta_{G/\Gamma}} = 1$, and therefore $\mu_{\Gamma\backslash G} = \mu_{G/\Gamma}$. Likewise, $\mu_{\Gamma\backslash G} = \mu_{G/\Gamma}$ when Γ is a closed subgroup in the center of G.

Example 2.2.9. Let G be the affine group on $\mathbb{R}$ consisting of transformations $x \mapsto ax + b$ with $a > 0$. As a topological space, we identify G with $(0, \infty) \times \mathbb{R}$; its group laws are then given by

$$(a,b)\cdot(c,d)=(ac,b+ad) \quad \text{and} \quad (a,b)^{-1}=(1/a,-b/a).$$

The modular function is $\Delta_G(a,b)=1/a$, and a left Haar measure is given by

$$d\mu_G(a,b)=\frac{da\,db}{a^2}.$$

Let H be the normal subgroup

$$H=\{(1,b)\in G\colon b\in\mathbb{R}\},$$

and let

$$K=\{(a,0)\in G\colon a>0\}.$$

Then $H\cong(\mathbb{R},+)$, $K\cong(\mathbb{R}_+,\times)$, and $G=H\rtimes K$. In particular, $G/H\cong K$. We choose Borel sections $\tau_H\colon G/H\to G$ and $\tau_K\colon G/K\to G$ given by

$$\tau_H((a,b)H)=(a,0) \quad \text{and} \quad \tau_K((a,b)K)=(1,b).$$

The associated sections $\gamma_H\colon H\backslash G\to G$ and $\gamma_K\colon K\backslash G\to G$ then have formulae

$$\gamma_H(H(a,b))=(a,0) \quad \text{and} \quad \gamma_K(K(a,b))=(1,b/a).$$

For Haar measures on H and K, we choose $d\mu_H(1,b)=db$ and $d\mu_K(a,0)=da/a$, respectively. Let us compute the measures on the respective quotients. Since H is a normal subgroup, $\mu_{G/H}$ is the unique choice of left Haar measure on $G/H\cong K\cong$

$(\mathbb{R}_+, \times)$ satisfying (2.15); an easy computation shows that

$$d\mu_{G/H}((a,b)H) = d\mu_K(a,0) = \frac{da}{a}.$$

Then by (2.19),

$$d\mu_{H\backslash G}(H(a,b)) = \frac{\Delta_G(\gamma_H(H(a,b)))}{\Delta_{G/H}((a,b)H)}\, d\mu_{G/H}(a,b) = \Delta_G(a,0)\,\frac{da}{a} = \frac{da}{a^2}.$$

On the other hand, K is not a normal subgroup, and since $\Delta_G(a,0) \neq \Delta_K(a,0)$ there is no invariant measure on G/K. However, a rho function is given by $\rho_K(a,b) = a$, and the associated quasi-invariant measure on $G/K \cong \mathbb{R}$ is

$$d\mu_{G/K}((a,b)K) = db,$$

as the reader can easily verify.

The interested reader can now compute $\mu_{K\backslash G}$ using (2.18). We proceed straight to the punchline. Since G/K is homeomorphic with $\mathbb{R}$ via $(1,b)K \mapsto b$, and since $(1,b)^{-1} = (1,-b)$, composing with $\varepsilon\colon K\backslash G \to G/K$ on the left and with $b \mapsto -b$ on the right shows that $K\backslash G \cong \mathbb{R}$ via $K(1,b) \mapsto b$. In particular, there is a positive regular Borel measure $\mu_{K\backslash G}$ on $K\backslash G$ given by

$$d\mu_{K\backslash G}(K(1,b)) = db.$$

For any $f \in C_c(G)$, we compute

$$
\begin{aligned}
\int_{K\backslash G} \int_K & f\Big((a,0) \cdot \gamma(K(1,b))\Big)\, d\mu_K(a,0)\, d\mu_{K\backslash G}(K(1,b)) \\
&= \int_{K\backslash G} \int_K f\big((a,0) \cdot (1,b)\big)\, d\mu_K(a,0)\, d\mu_{K\backslash G}(K(1,b)) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}_+} f(a,ab)\, \frac{da}{a}\, db = \int_{\mathbb{R}} \int_{\mathbb{R}_+} f(a,b) \frac{da\, db}{a^2} = \int_G f(a,b)\, d\mu_G(a,b).
\end{aligned}
$$

Therefore $\mu_{K\backslash G}$ satisfies (2.17). In other words, $K(1,b) \mapsto b$ identifies $K\backslash G$ with $\mathbb{R}$ as both a topological space and a measure space.

CHAPTER III

ACTIONS OF ABELIAN GROUPS

This chapter was previously published as [42, §4–6].

3.1. The Zak transform and fiberization

Let G be a second countable locally compact group with a closed *abelian* subgroup H, with notation as in the last section. In this section we develop a generalized version of the Zak transform for the pair (G, H).

We begin with a short reminder of terminology on locally compact abelian (LCA) groups. Let $\mathcal{G}$ be an LCA group, and let $\mu_{\mathcal{G}}$ be a Haar measure on $\mathcal{G}$. We denote $\hat{\mathcal{G}}$ for the dual group of $\mathcal{G}$, which consists of continuous homomorphisms $\alpha \colon \mathcal{G} \to \mathbb{T}$ under pointwise multiplication, with the topology of uniform convergence on compact sets. The *Fourier transform* of $f \in L^1(\mathcal{G})$ is the function $\hat{f} \in C_0(\hat{\mathcal{G}})$ given by

$$\hat{f}(\alpha) = \int_{\mathcal{G}} f(x)\, \alpha(x^{-1})\, d\mu_{\mathcal{G}}(x).$$

For any choice of Haar measure on $\hat{\mathcal{G}}$, the Fourier transform maps $L^1(\mathcal{G}) \cap L^2(\mathcal{G})$ onto a dense subspace of $L^2(\hat{\mathcal{G}})$, and for a unique choice $\mu_{\hat{\mathcal{G}}}$ this map is an isometry. That choice of $\mu_{\hat{\mathcal{G}}}$ is called *dual* to $\mu_{\mathcal{G}}$; it is always the measure we have in mind. With dual Haar measure on $\hat{\mathcal{G}}$, the Fourier transform extends uniquely to a unitary map $\mathcal{F}_{\mathcal{G}} \colon L^2(\mathcal{G}) \to L^2(\hat{\mathcal{G}})$. We also call $\mathcal{F}_{\mathcal{G}}$ the Fourier transform, and we also denote $\hat{f} = \mathcal{F}_{\mathcal{G}} f$ for $f \in L^2(\mathcal{G})$. When $g \in L^2(\hat{\mathcal{G}})$, we denote $\check{g} = \mathcal{F}_{\mathcal{G}}^{-1} g$. For any $f \in$

$L^1(\mathcal{G}) + L^2(\mathcal{G})$, the Fourier transform satisfies the following intertwining relation:

$$(L_y f)^\wedge(\alpha) = \alpha(y^{-1})\hat{f}(\alpha) \quad \text{for } y \in \mathcal{G},\, \alpha \in \hat{\mathcal{G}}.$$

The dual of $\hat{\mathcal{G}}$ can be identified with $\mathcal{G}$ as follows. Each $x \in \mathcal{G}$ defines a character X_x on $\hat{\mathcal{G}}$ given by $X_x(\alpha) = \alpha(x)$, and the mapping $x \mapsto X_x$ is a topological group isomorphism of $\mathcal{G}$ with $\hat{\hat{\mathcal{G}}}$. This is called Pontryagin Duality. When $\hat{\hat{\mathcal{G}}}$ is identified with $\mathcal{G}$ in this way, $\mu_{\mathcal{G}}$ gives the measure dual to $\mu_{\hat{\mathcal{G}}}$.

If $\Gamma \subseteq \mathcal{G}$ is a closed subgroup, its *annihilator* in $\hat{\mathcal{G}}$ is the closed subgroup

$$\Gamma^* = \{\kappa \in \hat{\mathcal{G}} : \kappa(\xi) = 1 \text{ for all } \xi \in \Gamma\}.$$

The subgroups $\Gamma \subseteq \mathcal{G}$, $\Gamma^* \subseteq \hat{\mathcal{G}}$, and their quotients are all canonically related through duality. First, each $\kappa \in \Gamma^*$ defines a character $\hat{\kappa} \in (\mathcal{G}/\Gamma)^\wedge$ by the formula

$$\hat{\kappa}(x\Gamma) = \kappa(x), \tag{3.1}$$

and the mapping $\kappa \mapsto \hat{\kappa}$ identifies Γ^* with $(\mathcal{G}/\Gamma)^\wedge$ as topological groups. Likewise, $\hat{\mathcal{G}}/\Gamma^*$ identifies with $\hat{\Gamma}$ through the mapping $\omega\Gamma^* \mapsto \omega|_\Gamma$. Moreover, the dual measures on $\hat{\mathcal{G}}$, $\Gamma^* \cong (\mathcal{G}/\Gamma)^\wedge$, and $\hat{\mathcal{G}}/\Gamma^* \cong \hat{\Gamma}$ satisfy Weil's formula (2.15).

Given $f\colon G \to \mathbb{C}$ and a coset $Hx \in H\backslash G$, we will write $f^{Hx}\colon H \to \mathbb{C}$ for the function

$$f^{Hx}(\xi) = f(\xi\gamma(Hx)),$$

where $\gamma\colon H\backslash G \to G$ is the Borel section from (2.16). Given $\varphi\colon \hat{H} \to L^2(H\backslash G)$, we define functions $\varphi_{Hx}\colon \hat{H} \to \mathbb{C}$ for a.e. $Hx \in H\backslash G$ with the formula

$$\varphi_{Hx}(\alpha) = \varphi(\alpha)(Hx).$$

Theorem 3.1.1. *There is a unitary transformation $Z\colon L^2(G) \to L^2(\hat{H}; L^2(H\backslash G))$ given by*

$$(Zf)(\alpha)(Hx) = \widehat{f^{Hx}}(\alpha) \qquad \text{for all } f \in L^2(G), \text{ a.e. } \alpha \in \hat{H}, \text{ and a.e. } Hx \in H\backslash G. \tag{3.2}$$

Its inverse is given by

$$(Z^{-1}\varphi)(\xi\gamma(Hx)) = \widetilde{\varphi_{Hx}}(\xi) \quad \text{for all } \varphi \in L^2(\hat{H}; L^2(H\backslash G)), \text{ a.e. } \xi \in H, \tag{3.3}$$

$$\text{and a.e. } Hx \in H\backslash G.$$

When $f \in L^2(G)$ and $\xi \in H$, Z satisfies the relation

$$(ZL_\xi f)(\alpha) = \alpha(\xi^{-1}) \cdot (Zf)(\alpha) \tag{3.4}$$

for a.e. $\alpha \in \hat{H}$.

Proof. Construct a sequence of unitaries

$$L^2(G) \xrightarrow{U_1} L^2(H \times H\backslash G) \xrightarrow{U_2} L^2(H\backslash G; L^2(H)) \xrightarrow{U_3} L^2(H\backslash G; L^2(\hat{H})) \xrightarrow{U_4} L^2(\hat{H}; L^2(H\backslash G)),$$

where U_1 is the isomorphism from Corollary 2.2.7, U_3 is the unitary given by

$$(U_3\varphi)(Hx) = \widehat{\varphi(Hx)},$$

and all others are the natural isomorphisms. Let $Z = U_4U_3U_2U_1$. Then

$$(U_2U_1f)(Hx)(\xi) = (U_1f)(\xi, Hx) = f(\xi\gamma(Hx)) = f^{Hx}(\xi),$$

and

$$(Zf)(\alpha)(Hx) = (U_4U_3U_2U_1f)(\alpha)(Hx) = (U_3U_2U_1f)(Hx)(\alpha) = [(U_2U_1f)(Hx)]^\wedge(\alpha)$$

$$= \widehat{f^{Hx}}(\alpha).$$

This proves (3.2). A similar computation verifies (3.3). Moreover, for every $f\colon G \to \mathbb{C}$ and every $\xi \in H$,

$$(L_\xi f)^{Hx}(\eta) = (L_\xi f)(\eta\gamma(Hx)) = f(\xi^{-1}\eta\gamma(Hx)) = f^{Hx}(\xi^{-1}\eta) = L_\xi(f^{Hx})(\eta).$$

Hence, for $f \in L^2(G)$ and $\xi \in H$,

$$(ZL_\xi f)(\alpha)(Hx) = [(L_\xi f)^{Hx}]^\wedge(\alpha) = [L_\xi(f^{Hx})]^\wedge(\alpha) = \alpha(\xi^{-1})\widehat{f^{Hx}}(\alpha)$$

$$= \alpha(\xi^{-1}) \cdot (Zf)(\alpha)(Hx),$$

as in (3.4). $\square$

We call Z the *Zak transform*, for reasons that will soon be obvious. Whenever we find it useful, we will freely interpret Z as the unitary $\tilde{Z}\colon L^2(G) \to L^2(\hat{H} \times H\backslash G)$

given by

$$(\tilde{Z}f)(\alpha, Hx) = (Zf)(\alpha)(Hx) = \widehat{f^{Hx}}(\alpha).$$

We emphasize that both Z *and* the measure used to construct $L^2(H\backslash G)$ depend on the choice of Borel section γ. Our construction of the Zak transform generalizes the definition given by Weil in [62, pp. 164–165] to the case where G is nonabelian; see Example 3.1.2(vi) below. For more on the history of the Zak transform, we refer the reader to [39].

Example 3.1.2. We now compute Z in a wide variety of concrete settings.

(i) $\mathbb{Z} \subseteq \mathbb{R}$. To justify our usage of "Zak transform", we first compute Z for the subgroup $\mathbb{Z} \subseteq \mathbb{R}$. Take Lebesgue measure for $\mu_{\mathbb{R}}$ and counting measure for $\mu_{\mathbb{Z}}$. We use the fundamental domain $[0,1)$. Since $\mathbb{Z}$ is discrete, the associated section γ identifies $(\mathbb{Z}\backslash\mathbb{R}, \mu_{\mathbb{Z}\backslash\mathbb{R}})$ with the interval $[0,1)$ under Lebesgue measure, as explained in Remark 2.2.5. From this perspective, $f^{t+\mathbb{Z}}(k) = f(t+k)$ for $f\colon \mathbb{R} \to \mathbb{C}$, $t \in [0,1)$, and $k \in \mathbb{Z}$. When $\hat{\mathbb{Z}}$ is identified with $\mathbb{T}$, the Zak transform becomes the map $Z\colon L^2(\mathbb{R}) \to L^2(\mathbb{T}; L^2([0,1)))$ which for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is given by

$$(Zf)(z)(t) = \widehat{f^{t+\mathbb{Z}}}(z) = \sum_{k\in\mathbb{Z}} f^{t+\mathbb{Z}}(k) z^{-k} = \sum_{k\in\mathbb{Z}} f(t+k) z^{-k}.$$

If we further identify $\mathbb{T}$ with the interval $[0,1)$ under Lebesgue measure, Z can be thought of as the map $\tilde{Z}\colon L^2(\mathbb{R}) \to L^2([0,1)\times[0,1))$ given by

$$(\tilde{Z}f)(s,t) = \sum_{k\in\mathbb{Z}} f(t+k) e^{-2\pi i k s} \tag{3.5}$$

for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $s,t \in [0,1)$. This is exactly the classical Zak transform.

(ii) $\mathbb{Z}^m \subseteq \mathbb{R}^n$. More generally, let m and n be positive integers with $m \leq n$, and think of $\mathbb{Z}^m$ as the subgroup of $\mathbb{R}^n$ consisting of vectors with integers in the first m entries and zeros in the last $n-m$. A fundamental domain is given by $[0,1)^m \times \mathbb{R}^{n-m}$, and since $\mathbb{Z}^m$ is discrete, the associated section $\gamma\colon \mathbb{Z}^m \backslash \mathbb{R}^n \to \mathbb{R}^n$ identifies the measure space $(\mathbb{Z}^m \backslash \mathbb{R}^n, \mu_{\mathbb{Z}^m \backslash \mathbb{R}^n})$ with $[0,1)^m \times \mathbb{R}^{n-m}$ under Lebesgue measure. Identifying $\widehat{\mathbb{Z}^m}$ with $[0,1)^m \subseteq \mathbb{R}^m$ as above, we can think of the Zak transform as a unitary

$$\tilde{Z}\colon L^2(\mathbb{R}^n) \to L^2([0,1)^m \times [0,1)^m \times \mathbb{R}^{n-m})$$

which for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is given by

$$(\tilde{Z}f)(s,t,x) = \sum_{k \in \mathbb{Z}^m} f(t+k,x) e^{-2\pi i k \cdot s}. \tag{3.6}$$

(iii) $\mathbb{R}^m \subseteq \mathbb{R}^n$. Let m and n be positive integers with $m \leq n$, and consider $\mathbb{R}^m$ as the subgroup of $\mathbb{R}^n$ consisting of vectors with zeros in the last $n-m$ entries. Then $\mathbb{R}^m \backslash \mathbb{R}^n \cong \mathbb{R}^{n-m}$ with Lebesgue measure, by (2.13) and Remark 2.2.8. Our section $\gamma\colon \mathbb{R}^{n-m} \to \mathbb{R}^n$ will be given by

$$\gamma(x_{m+1}, \ldots, x_n) = (0, \ldots, 0, x_{m+1}, \ldots, x_n).$$

Identifying $\hat{\mathbb{R}^m}$ with $\mathbb{R}^m$ in the usual way, we can view the Zak transform for $\mathbb{R}^m \subseteq \mathbb{R}^n$ as a unitary

$$Z\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m; L^2(\mathbb{R}^{n-m}))$$

which for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is given by

$$(Zf)(\xi)(y) = \int_{\mathbb{R}^m} f(x,y)e^{-2\pi i \xi \cdot x}\,dx. \tag{3.7}$$

(iv) $\mathbb{Z}_p \subseteq \mathbb{Q}_p$. Let p be a prime number, and let $\mathbb{Q}_p$ be the locally compact field of p-adic numbers

$$x = \sum_{j=m}^{\infty} c_j p^j$$

for $m \in \mathbb{Z}$ and $c_j \in \{0, 1, \ldots, p-1\}$. The topology on $\mathbb{Q}_p$ is given by the p-adic norm $|\cdot|_p$; for x as above with $c_m \neq 0$, $|x|_p = p^{-m}$. Any two elements of $\mathbb{Q}_p$ can be added or multiplied in the obvious way, and under these operations $\mathbb{Q}_p$ is a locally compact field. Consider $\mathbb{Q}_p$ as an LCA group under addition, and let $\mathbb{Z}_p$ be the compact open subgroup of p-adic integers

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \left\{ \sum_{j=0}^{\infty} c_j p^j : c_j \in \{0, 1, \ldots, p-1\} \right\}.$$

A fundamental domain for $\mathbb{Z}_p$ is

$$\Omega = \left\{ \sum_{j=m}^{-1} c_j p^j : m \in \mathbb{Z}_{<0},\ c_j \in \{0, 1, \ldots, p-1\} \right\}.$$

Since $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$, the quotient $\mathbb{Z}_p \backslash \mathbb{Q}_p$ is discrete, and the section $\gamma \colon \mathbb{Z}_p \backslash \mathbb{Q}_p \to \mathbb{Q}_p$ associated with Ω is automatically Borel. Moreover, the image of a compact set automatically has compact closure, as required.

Identify $\hat{\mathbb{Q}}_p$ with $\mathbb{Q}_p$ as follows. For $x = \sum_{j=m}^{\infty} c_j p^j \in \mathbb{Q}_p$, we abbreviate

$$e^{\pm 2\pi i x} = \exp\left(\pm 2\pi i \sum_{j=m}^{-1} c_j p^j \right).$$

Each $y \in \mathbb{Q}_p$ then defines a character $\omega_y \in \hat{\mathbb{Q}}_p$ by the formula $\omega_y(x) = e^{2\pi i y x}$, and the mapping $y \mapsto \omega_y$ is a topological group isomorphism of $\mathbb{Q}_p$ with $\hat{\mathbb{Q}}_p$. Moreover,

$$\mathbb{Z}_p^* = \{\omega_\xi : \xi \in \mathbb{Z}_p\}.$$

Hence $\hat{\mathbb{Z}}_p \cong \hat{\mathbb{Q}}_p/\mathbb{Z}_p^*$ is the discrete group of characters $\omega_y|_{\mathbb{Z}_p}$ for $y \in \Omega$.

When Haar measures are normalized so that $\mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = \mu_{\mathbb{Z}_p}(\mathbb{Z}_p) = 1$, the dual measure on $\hat{\mathbb{Z}}_p$ is counting measure. Counting measure on $\mathbb{Z}_p \backslash \mathbb{Q}_p$ also causes (2.17) to hold. Identifying both $\hat{\mathbb{Z}}_p$ and $\mathbb{Z}_p \backslash \mathbb{Q}_p$ with Ω makes the Zak transform a unitary

$$\tilde{Z} \colon L^2(\mathbb{Q}_p) \to l^2(\Omega \times \Omega)$$

which for $f \in L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ is given by

$$(\tilde{Z} f)(x, y) = \int_{\mathbb{Z}_p} f(y + \xi) e^{-2\pi i x \xi} \, d\mu_{\mathbb{Z}_p}(\xi). \tag{3.8}$$

(v) $\mathbb{R} \subseteq ax + b$. Let G be the $ax + b$ group described in Example 2.2.9. For the normal subgroup

$$H = \{(1, b) \in G : b \in \mathbb{R}\} \cong (\mathbb{R}, +),$$

we identify $\hat{H}$ with $\mathbb{R}$ in the usual way. When $L^2(H \backslash G)$ is identified with $L^2(\mathbb{R}_+, dx/x^2)$ via $x \mapsto H(x, 0)$, the Zak transform becomes a unitary $Z_H \colon L^2(G) \to L^2(\mathbb{R}; L^2(\mathbb{R}_+, dx/x^2))$ which for $f \in L^1(G) \cap L^2(G)$ is given by

$$(Z_H f)(\xi)(a) = \int_{\mathbb{R}} f(a, b) e^{-2\pi i \xi b} \, db.$$

On the other hand, the subgroup

$$K = \{(a, 0) \in G : a > 0\}$$

is isomorphic with $(\mathbb{R}_+, \times)$, and its dual can be identified with $(\mathbb{R}, +)$ under the pairing

$$\hat{\xi}(a, 0) = a^{2\pi i \xi}.$$

For $(a, 0) \in K$ and $K(1, b) \in K \backslash G$, we have

$$f^{K(1,b)}(a, 0) = f\big((a, 0) \cdot \gamma_K(K(1, b))\big) = f\big((a, 0) \cdot (1, b)\big) = f(a, ab).$$

Identifying $L^2(K \backslash G)$ with $L^2(\mathbb{R})$ as in Example 2.2.9, the Zak transform becomes a unitary $Z_K \colon L^2(G) \to L^2(\mathbb{R} \times \mathbb{R})$ which for $f \in L^1(G) \cap L^2(G)$ is given by

$$(Z_K f)(\xi, b) = \int_0^\infty \frac{f(a, ab) a^{-2\pi i \xi}}{a} \, da.$$

(vi) Let G be any second countable LCA group with closed subgroup H. For $f \in C_c(G)$, $\alpha \in \hat{H}$, and $Hx \in H \backslash G$, we compute

$$(\tilde{Z} f)(\alpha, Hx) = \widehat{f^{Hx}}(\alpha) = \int_H f(\xi \gamma(Hx)) \alpha(\xi^{-1}) \, d\mu_H(\xi).$$

More generally, for $x \in G$ the function $\xi \mapsto f(\xi x)$ belongs to $C_c(H)$, so we can define a function $\tilde{\tilde{Z}} f \colon G \times \hat{G} \to \mathbb{C}$ by the formula

$$(\tilde{\tilde{Z}} f)(\omega, x) = \int_H f(\xi x) \omega(\xi^{-1}) \, d\mu_H(\xi). \tag{3.9}$$

This matches the definition of Zak transform given by Weil in [62] (although he didn't call it that, of course).

For the remainder of this section, we will assume that G is abelian. Each $\omega \in \hat{G}$ then acts unitarily on $L^2(G)$ via the modulation $(M_\omega f)(x) = \omega(x)f(x)$. The Zak transform behaves well under modulations by H^* and translations by H. When $f \in C_c(G)$ and $\kappa \in H^*$,

$$(\tilde{Z}M_\kappa f)(\alpha, Hx) = \int_H (M_\kappa f)(\xi\gamma(Hx))\alpha(\xi^{-1})\, d\mu_H(\xi)$$

$$= \int_H \kappa(\xi\gamma(Hx))f(\xi\gamma(Hx))\alpha(\xi^{-1})\, d\mu_H(\xi)$$

$$= \int_H \kappa(\gamma(Hx))f(\xi\gamma(Hx))\alpha(\xi^{-1})\, d\mu_H(\xi) = \kappa(\gamma(Hx)) \cdot (\tilde{Z}f)(\alpha, Hx).$$

Since $\gamma(Hx) = \xi x$ for some $\xi \in H$, and since $\kappa(\xi) = 1$, we can write $\kappa(x)$ in place of $\kappa(\gamma(Hx))$ in the last expression above. Extending by continuity and combining with (3.4), we find that

$$(\tilde{Z}L_\xi M_\kappa f)(\alpha, Hx) = \alpha(\xi^{-1})\kappa(x) \cdot (\tilde{Z}f)(\alpha, Hx) \tag{3.10}$$

for all $f \in L^2(G)$, $\xi \in H$, and $\kappa \in H^*$.

In the abelian setting, the Zak transform has a sibling, which we now introduce. Whenever we work in this setting we will use a fixed Borel section $\beta \colon \hat{G}/H^* \to \hat{G}$ that sends compact sets to pre-compact sets. We remind the reader that $G/H = H\backslash G$ and $\hat{G}/H^* = H^*\backslash\hat{G}$ as measure spaces; see the final line of Remark 2.2.8.

Proposition 3.1.3. *In addition to the standing hypotheses, suppose that G is abelian. There is a unitary map*

$$\mathcal{T}\colon L^2(G) \to L^2(\hat{G}/H^*; L^2(H^*))$$

given by

$$(\mathcal{T}f)(\omega H^*)(\kappa) = \hat{f}(\beta(\omega H^*)\kappa). \tag{3.11}$$

Moreover, for any $\xi \in H$,

$$(\mathcal{T}L_\xi f)(\omega H^*) = \omega(\xi^{-1}) \cdot (\mathcal{T}f)(\omega H^*). \tag{3.12}$$

Proof. Follow the Fourier transform $L^2(G) \to L^2(\hat{G})$ by the unitary $L^2(\hat{G}) \to L^2(H^* \times \hat{G}/H^*)$ from Corollary 2.2.3. When $L^2(H^* \times \hat{G}/H^*)$ is identified with $L^2(\hat{G}/H^*; L^2(H^*))$, the composition $\mathcal{T}$ is given by (3.11).

If $f \in L^2(G)$ and $\xi \in H$, we compute

$$(\mathcal{T}L_\xi f)(\omega H^*)(\kappa) = (L_\xi f)^\wedge(\beta(\omega H^*)\kappa) = \beta(\omega H^*)(\xi^{-1})\kappa(\xi^{-1})\hat{f}(\beta(\omega H^*)\kappa)$$

$$= \omega(\xi^{-1}) \cdot (\mathcal{T}f)(\omega H^*)(\kappa),$$

since $\beta(\omega H^*) = \omega\chi$ for some $\chi \in H^*$, and $\chi(\xi^{-1}) = \kappa(\xi^{-1}) = 1$. This proves (3.12). $\square$

We call $\mathcal{T}$ the *fiberization* map. In the special case where H is discrete and G/H is compact, Proposition 3.1.3 was proved separately by Kamyabi Gol and Raisi Tousi [49, Proposition 2.1] and Cabrelli and Paternostro [14, Proposition 3.3]. To the author's knowledge, every existing classification of H-TI uses some version of fiberization.

The Zak transform is closely related to fiberization in the abelian setting, and indeed $\mathcal{T}$ can be obtained from Z through a modulation in $L^2(\hat{H}; L^2(H\backslash G))$ and the Fourier transform on $H\backslash G$, as we now show. With the isomorphisms $\hat{H} \cong \hat{G}/H^*$ and $H^* \cong (G/H)^\wedge$ in mind, define a modulation $M\colon L^2(\hat{H}; L^2(H\backslash G)) \to L^2(\hat{H}; L^2(H\backslash G))$ by the formula

$$(M\varphi)(\omega|_H)(Hx) = \beta(\omega H^*)(\gamma(Hx)^{-1}) \cdot \varphi(\omega|_H)(Hx) \tag{3.13}$$

for $\varphi \in L^2(\hat{H}; L^2(H\backslash G))$, $\omega \in \hat{G}$, and $Hx \in H\backslash G$. We claim that

$$(\mathcal{T}f)(\omega H^*)(\kappa) = [(MZf)(\omega|_H)]^\wedge(\hat{\kappa}) \tag{3.14}$$

for any $f \in L^2(G)$, where the Fourier transform on the right is taken over $H\backslash G$. Indeed, for any $f \in C_c(G)$, we compute

$$(\mathcal{T}f)(\omega H^*)(\kappa) = \hat{f}(\beta(\omega H^*)\kappa) = \int_G f(x)\beta(\omega H^*)(x^{-1})\kappa(x^{-1})\, d\mu_G(x)$$

$$= \int_{H\backslash G}\int_H f(\xi\gamma(Hx)) \cdot \beta(\omega H^*)(\gamma(Hx)^{-1}\xi^{-1}) \cdot \kappa(\gamma(Hx)^{-1}\xi^{-1})\, d\mu_H(\xi)\, d\mu_{H\backslash G}(Hx)$$

$$= \int_{H\backslash G} \beta(\omega H^*)(\gamma(Hx)^{-1}) \cdot \kappa(\gamma(Hx)^{-1}) \int_H f(\xi\gamma(Hx)) \cdot \beta(\omega H^*)(\xi^{-1}) \cdot \kappa(\xi^{-1})\, d\mu_H(\xi)\, d\mu_{H\backslash G}(Hx).$$

This is messy, but it cleans up nicely. First, $\kappa(\gamma(Hx)^{-1}) = \kappa(x^{-1})$, since $\kappa(\eta) = 1$ for any $\eta \in H$. Likewise, $\beta(\omega H^*)(\xi^{-1}) = \omega(\xi^{-1})$, since any element of H^* annihilates ξ^{-1}. We also have $\kappa(\xi^{-1}) = 1$, and we can abbreviate $f(\xi\gamma(Hx)) = f^{Hx}(\xi)$. With all that in mind, our last equation reads

$$(\mathcal{T}f)(\omega H^*)(\kappa) = \int_{H\backslash G} \beta(\omega H^*)(\gamma(Hx)^{-1}) \cdot \kappa(x^{-1}) \int_H f^{Hx}(\xi)\omega(\xi^{-1})\, d\mu_H(\xi)\, d\mu_{H\backslash G}(Hx)$$

$$= \int_{H\backslash G} \beta(\omega H^*)(\gamma(Hx)^{-1}) \cdot \widehat{f^{Hx}}(\omega|_H) \cdot \kappa(x^{-1})\, d\mu_{H\backslash G}(Hx)$$

$$= \int_{H\backslash G} (MZf)(\omega|_H)(Hx) \cdot \hat{\kappa}(Hx^{-1})\, d\mu_{H\backslash G}(Hx) = [(MZf)(\omega|_H)]^\wedge(\hat{\kappa}).$$

Thus (3.14) holds for all $f \in C_c(G)$; extending with continuity gives it for all $f \in L^2(G)$.

Example 3.1.4. Let us interpret (3.14) for the classical Zak transform (3.5). Identify $\hat{\mathbb{R}}$ with $\mathbb{R}$ and $\mathbb{Z}^*$ with $\mathbb{Z}$ in the usual way: each $\xi \in \mathbb{R}$ defines a character $\hat{\xi} \in \hat{\mathbb{R}}$ by $\hat{\xi}(x) = e^{2\pi i \xi x}$, and $\mathbb{Z}^* = \{\hat{k} \in \hat{\mathbb{R}} : k \in \mathbb{Z}\}$. In Example 3.1.2(i), the identification of $\mathbb{T} \cong \hat{\mathbb{Z}} \cong \hat{\mathbb{R}}/\mathbb{Z}^*$ with $[0,1) \subseteq \mathbb{R}$ describes a Borel section $\beta\colon \hat{\mathbb{R}}/\mathbb{Z}^* \to \hat{\mathbb{R}}$ with fundamental domain $\beta(\hat{\mathbb{R}}/\mathbb{Z}^*) = [0,1) \subseteq \hat{\mathbb{R}}$. Then for $\varphi \in L^2([0,1) \times [0,1))$, the modulation M in (3.13) is given by

$$(M\varphi)(s,t) = \hat{s}(-t)\varphi(s,t) = e^{-2\pi i s t}\varphi(s,t).$$

Thus, (3.14) says that for all $f \in L^2(\mathbb{R})$ and a.e. $s \in [0,1)$,

$$\hat{f}(s+k) = \int_0^1 e^{-2\pi i s t} \cdot (\tilde{Z}f)(s,t) \cdot e^{-2\pi i k t}\, dt = \int_0^1 (\tilde{Z}f)(s,t) \cdot e^{-2\pi i (s+k)t}\, dt \quad \text{for all } k \in \mathbb{Z}.$$

Here is another relation between the fiberization map and the Zak transform. Fix $f, g \in L^2(G)$. For every $\xi \in H$, (3.4) and (2.12) show that

$$\langle f, L_\xi g\rangle = \langle Zf, Z(L_\xi g)\rangle = \int_{\hat{H}} \langle (Zf)(\alpha), (ZL_\xi g)(\alpha)\rangle d\mu_{\hat{H}}(\alpha) \tag{3.15}$$

$$= \int_{\hat{H}} \langle (Zf)(\alpha), (Zg)(\alpha) \rangle \alpha(\xi)\, d\mu_{\hat{H}}(\alpha)$$

$$= \int_{\hat{H}} \langle (Zf)(\alpha^{-1}), (Zg)(\alpha^{-1}) \rangle \overline{\alpha(\xi)}\, d\mu_{\hat{H}}(\alpha).$$

On the other hand, a similar computation involving (3.12) produces

$$\langle f, L_\xi g \rangle = \int_{\hat{G}/H^*} \langle (\mathcal{T}f)(\omega^{-1}H^*), (\mathcal{T}g)(\omega^{-1}H^*) \rangle \overline{\omega(\xi)}\, d\mu_{\hat{G}/H*}(\omega H^*) \tag{3.16}$$

$$= \int_{\hat{H}} \langle (\mathcal{T}f)(\omega^{-1}H^*), (\mathcal{T}g)(\omega^{-1}H^*) \rangle \overline{\omega(\xi)}\, d\mu_{\hat{H}}(\omega|_H).$$

The Fourier transform $L^1(\hat{H}) \to C_0(H)$ is injective, so for a.e. $\omega H^* \in \hat{G}/H^*$,

$$\langle (\mathcal{T}f)(\omega H^*), (\mathcal{T}g)(\omega H^*) \rangle_{L^2(H^*)} = \langle (Zf)(\omega|_H), (Zg)(\omega|_H) \rangle_{L^2(H \backslash G)}. \tag{3.17}$$

3.2. The structure of H-TI spaces in $L^2(G)$

Returning to the more general case, where G need not be abelian, we now classify H-TI spaces in $L^2(G)$. Given a family $\mathscr{A} \subseteq L^2(G)$, we will denote

$$E^H(\mathscr{A}) = \{ L_\xi \varphi : \xi \in H, \varphi \in \mathscr{A} \}$$

for the left H-translates of $\mathscr{A}$, and

$$S^H(\mathscr{A}) = \overline{\operatorname{span}} \{ L_\xi \varphi : \xi \in H, \varphi \in \mathscr{A} \}$$

for the H-TI space it generates. We will also give conditions under which $E^H(\mathscr{A})$ forms a continuous frame or a Riesz basis for $S^H(\mathscr{A})$.

When $J\colon \hat{H} \to \{\text{closed subspaces of } L^2(H\backslash G)\}$ is a range function, we write $P_J(\alpha)\colon L^2(H\backslash G) \to J(\alpha)$ for the orthogonal projection associated to $\alpha \in \hat{H}$. We also denote

$$V_J = \{f \in L^2(G) : (Zf)(\alpha) \in J(\alpha) \text{ for a.e. } \alpha \in \hat{H}\}.$$

If G is abelian and $\tilde{J}\colon \hat{G}/H^* \to \{\text{closed subspaces of } L^2(H^*)\}$, we similarly write $\tilde{P}_{\tilde{J}}(\omega H^*)\colon L^2(H^*) \to \tilde{J}(\omega H^*)$ for the orthogonal projection associated to $\omega H^* \in \hat{G}/H^*$, and we define

$$\tilde{V}_{\tilde{J}} = \{f \in L^2(G) : (\mathcal{T}f)(\omega H^*) \in \tilde{J}(\omega H^*) \text{ for a.e. } \omega H^* \in \hat{G}/H^*\}.$$

The next theorem is an application of [13, Theorem 2.4]. Its provenance stretches back to Helson [37] and Srinivasan [57]. Part (ii) generalizes results of de Boor, DeVore, and Ron [22, Result 1.5]; Bownik [11, Proposition 1.5]; Cabrelli and Paternostro [14, Theorem 3.10]; Kamyabi Gol and Raisi Tousi [49, Theorem 3.1]; and Bownik and Ross [13, Theorem 3.8]. In contrast with these references, we do not require G/H to be compact. Part (i) opens the door even wider, by allowing G to be nonabelian. As far as the author knows, the results in (i) are new even for $\mathbb{Z} \subseteq \mathbb{R}$.

For another description of H-TI spaces, in terms of the "extra" invariance of an invariant subspace, we refer the reader to [1, 3, 4, 56]. In the special case where G is abelian and H contains a countable discrete subgroup K such that G/K is compact, these papers describe H-TI spaces in terms of the range function classification of K-TI spaces given in [11, 14, 22, 49]. In particular, their descriptions of H-TI spaces use the fiberization map for $K \subseteq G$. We do not require H to contain such a subgroup here, and our classifications are in terms of the Zak transform and fiberization map for H itself.

Theorem 3.2.1. *(i) H-TI spaces in $L^2(G)$ are indexed by measurable range functions*

$$J\colon \hat{H} \to \{\textit{closed subspaces of } L^2(H\backslash G)\},$$

provided we identify range functions that agree a.e. A bijection maps $J \mapsto V_J$. When $\mathscr{A} \subseteq L^2(G)$ is a family with a countable dense subset $\mathscr{A}_0 \subseteq \mathscr{A}$, $S^H(\mathscr{A}) = S^H(\mathscr{A}_0)$, and the associated range function is given by

$$J(\alpha) = \overline{\operatorname{span}}\{(Zf)(\alpha) : f \in \mathscr{A}_0\}. \tag{3.18}$$

(ii) In addition to the standing assumptions, suppose that G is abelian. Then H-TI spaces in $L^2(G)$ can also be indexed by measurable range functions

$$\tilde{J}\colon \hat{G}/H^* \to \{\textit{closed subspaces of } L^2(H^*)\},$$

provided we identify range functions that agree a.e. A bijection maps $\tilde{J} \mapsto \tilde{V}_{\tilde{J}}$. For a family $\mathscr{A} \subseteq L^2(G)$ with countable dense subset $\mathscr{A}_0 \subseteq \mathscr{A}$, the range function associated with $S^H(\mathscr{A}) = S^H(\mathscr{A}_0)$ is

$$\tilde{J}(\omega H^*) = \overline{\operatorname{span}}\{(\mathcal{T}f)(\omega H^*) : f \in \mathscr{A}_0\}. \tag{3.19}$$

We will need the following lemma, which essentially restates [13, Lemma 3.5].

Lemma 3.2.2. *Let $\mathcal{G}$ be an LCA group with Haar measure $\mu_{\mathcal{G}}$, and let $\hat{\mathcal{G}}$ be its dual group with dual Haar measure $\mu_{\hat{\mathcal{G}}}$. Then $\hat{\mathcal{G}}$ forms a Parseval determining set for*

$L^1(\mathcal{G})$ with respect to $\mu_{\hat{\mathcal{G}}}$. In other words,

$$\int_{\hat{\mathcal{G}}} \left| \int_{\mathcal{G}} f(x)\overline{\alpha(x)}\, d\mu_{\mathcal{G}}(x) \right|^2 d\mu_{\hat{\mathcal{G}}}(\alpha) = \int_{\mathcal{G}} |f(x)|^2\, d\mu_{\mathcal{G}}(x) \tag{3.20}$$

for each $f \in L^1(\mathcal{G})$; both sides may be infinite.

Proof. For $f \in L^1(\mathcal{G})$, the left hand side of (3.20) is precisely $\left\|\hat{f}\right\|_2^2$. If $f \in L^1(\mathcal{G}) \cap L^2(\mathcal{G})$, (3.20) is just Plancherel's Theorem. On the other hand, if $\|f\|_2 = \infty$, then $\left\|\hat{f}\right\|_2 = \infty$ by 31.44(a) of [41]. $\square$

Remark 3.2.3. Pontryagin Duality allows us to switch $\mathcal{G}$ and $\hat{\mathcal{G}}$ in the lemma above. Given $x \in \mathcal{G}$, write $X_x \in \hat{\hat{\mathcal{G}}}$ for the corresponding character $X_x(\alpha) = \alpha(x)$. Then $\mathcal{D} = (X_x)_{x\in\mathcal{G}}$ is a Parseval determining set for $L^1(\hat{\mathcal{G}})$ with respect to $\mu_{\mathcal{G}}$.

When G is abelian, we can identify $(\hat{G}/H^*, \mu_{\hat{G}/H^*})$ with $(\hat{H}, \mu_{\hat{H}})$ by mapping $\omega H^* \mapsto \omega|_H$. Each $\xi \in H$ then defines a character $\tilde{X}_\xi$ on $\hat{G}/H^*$ by the formula

$$\tilde{X}_\xi(\omega H^*) = X_\xi(\omega|_H) = \omega(\xi),$$

and the previous paragraph shows that $\tilde{\mathcal{D}} = (\tilde{X}_\xi)_{\xi\in H}$ is a Parseval determining set for $L^1(\hat{G}/H^*)$ with respect to μ_H.

Proof of Theorem 3.2.1. Let $\mathcal{D}$ be as in Remark 3.2.3, with $\mathcal{G} = H$. By Theorem 3.1.1, a subspace $M \subseteq L^2(G)$ is H-TI if and only if ZM is a $\mathcal{D}$-MI subspace of $L^2(\hat{H}; L^2(H\backslash G))$. Thus (i) is an application of Proposition 2.1.2. Likewise, (ii) follows immediately from Proposition 3.1.3, Proposition 2.1.2, and the remark above. $\square$

As in the familiar case of integer shifts in $L^2(\mathbb{R}^n)$, our classification of H-TI spaces ties with a set of conditions under which the H-translates of a family $\mathscr{A} \subseteq L^2(G)$ form a continuous frame. Namely, it reduces the problem of $E^H(\mathscr{A})$ forming a continuous frame for $S^H(\mathscr{A})$ to an analysis of the fibers $J(\alpha) = \overline{\operatorname{span}}\{(Zf)(\alpha) : f \in \mathscr{A}\}$. If G is abelian we can replace the Zak transform with fiberization, and if H is discrete we can replace "continuous frame" with "Riesz basis".

The next two theorems are applications of Theorems 2.1.3 and 2.1.10. They generalize results of Bownik [11, Theorem 2.3]; Kamyabi Gol and Raisi Tousi [49, Theorems 4.1 and 4.2]; Cabrelli and Paternostro [14, Theorems 4.1 and 4.3]; and Bownik and Ross [13, Theorem 5.1]. In contrast with these results, we do not require G/H to be compact. When we use the Zak transform, we do not even need G to be abelian.

Theorem 3.2.4. *Let $(\mathcal{M}, \mu_{\mathcal{M}})$ be a complete, σ-finite measure space, and let $\mathscr{A} = (f_t)_{t\in\mathcal{M}} \subseteq L^2(G)$ be a jointly measurable family of functions. Fix a countable dense subset $\mathscr{A}_0 \subseteq \mathscr{A}$, and let J be as in* (3.18)*. Given constants $0 < A \leq B < \infty$, the following are equivalent:*

(i) $E^H(\mathscr{A})$ forms a continuous frame for $S^H(\mathscr{A})$ over $\mathcal{M} \times H$, with bounds A, B. In other words, for every $g \in S^H(\mathscr{A})$,

$$A \int_G |g(x)|^2 \, d\mu_G(x) \leq \int_{\mathcal{M}} \int_H \left| \int_G g(x) \overline{L_\xi f_t(x)} \, d\mu_G(x) \right|^2 d\mu_H(\xi) \, d\mu_{\mathcal{M}}(t)$$

$$\leq B \int_G |g(x)|^2 \, d\mu_G(x).$$

(ii) For a.e. $\alpha \in \hat{H}$ and every $h \in J(\alpha) \subseteq L^2(H \backslash G)$,

$$A \|h\|^2 \leq \int_{\mathcal{M}} |\langle h, (Zf_t)(\alpha) \rangle|^2 \, d\mu_{\mathcal{M}}(t) \leq B \|h\|^2 .$$

If G is abelian and $\tilde{J}$ is as in (3.19)*, the conditions above are equivalent to:*

(iii) For a.e. $\omega H^* \in \hat{G}/H^*$ *and every* $h \in \tilde{J}(\omega H^*) \subseteq L^2(H^*)$,

$$A \left\| h \right\|^2 \leq \int_{\mathcal{M}} \left| \langle h, (\mathcal{T} f_t)(\omega H^*) \rangle \right|^2 \, d\mu_{\mathcal{M}}(t) \leq B \left\| h \right\|^2 .$$

As in the remarks following Theorem 2.1.10, condition (ii) says that for a.e. $\alpha \in \hat{H}$, the family $\{[P_J(\alpha)](Zf_t)(\alpha) : t \in \mathcal{M}\}$ forms a continuous frame for $J(\alpha)$ with bounds A, B. A similar consideration applies to (iii).

When $\mathscr{A}$ is countable this theorem reduces a continuous problem in $L^2(G)$ to a discrete problem in $L^2(H \backslash G)$ or $L^2(H^*)$. For instance, when $\mathscr{A}$ consists of a single function $f \in L^2(G)$, condition (ii) is equivalent to

(ii') For a.e. $\alpha \in \hat{H}$, either $(Zf)(\alpha) = 0$ or $A \leq \left\| (Zf)(\alpha) \right\|^2 \leq B$.

Proof of Theorem 3.2.4. We claim that $Z\mathscr{A} = (Zf_t)_{t \in \mathcal{M}} \subseteq L^2(\hat{H}; L^2(H \backslash G))$ is jointly measurable. To prove this, we consider the image of $\mathscr{A}$ under each of the isomorphisms U_k used to construct Z in the proof of Theorem 3.1.1. The first isomorphism $U_1 \colon L^2(G) \to L^2(H \times H \backslash G)$ is gotten from a measure space isomorphism, so it must preserve the notion of joint measurability. Corollary 2.1.9 shows joint measurability is preserved by $U_2 \colon L^2(H \times H \backslash G) \to L^2(H \backslash G; L^2(H))$. Since the Fourier transform $L^2(H) \to L^2(\hat{H})$ leaves inner products unchanged, joint measurability is preserved by $U_3 : L^2(H \backslash G; L^2(H)) \to L^2(H \backslash G; L^2(\hat{H}))$. Another application of Corollary 2.1.9 gives joint measurability after applying $U_4 \colon L^2(H \backslash G; L^2(\hat{H})) \to L^2(\hat{H}; L^2(H \backslash G))$. This proves the claim.

Let $\mathcal{D} = (X_\xi)_{\xi \in H}$ be the Parseval determining set from Remark 3.2.3. Since the unitary $Z \colon L^2(G) \to L^2(\hat{H}; L^2(H \backslash G))$ intertwines left translation by $\xi \in H$ with multiplication by $X_\xi \in \mathcal{D}$, condition (i) above is equivalent to:

(i') $E_{\mathcal{D}}(Z\mathscr{A})$ forms a continuous frame for $S_{\mathcal{D}}(Z\mathscr{A})$ with bounds A, B.

Moreover, the range function associated with the $\mathcal{D}$-MI space $S_{\mathcal{D}}(Z\mathscr{A})$ is precisely J, by Proposition 2.1.2. Hence, the equivalence of (i) and (ii) follows from the corresponding equivalence in Theorem 2.1.10.

When G is abelian, the fiberization map $\mathcal{T}$ is made by composing the Fourier transform $L^2(G) \to L^2(\hat{G})$ with the isomorphisms $L^2(\hat{G}) \to L^2(H^* \times \hat{G}/H^*)$ and $L^2(H^* \times \hat{G}/H^*) \to L^2(\hat{G}/H^*; L^2(H^*))$. The first isomorphism preserves joint measurability as an easy consequence of Proposition 2.1.8 and Plancherel's Theorem, the second preserves it because it is based on a measure space isomorphism, and the third preserves it by Corollary 2.1.9. Consequently, $\mathcal{T}\mathscr{A} = (\mathcal{T}f_t)_{t\in\mathcal{M}} \subseteq L^2(\hat{G}/H^*; L^2(H^*))$ is jointly measurable. An argument similar to the one in the paragraph above now proves the equivalence of (i) and (iii): replace Z with $\mathcal{T}$, and $\mathcal{D}$ with $\tilde{\mathcal{D}}$ from Remark 3.2.3. □

Theorem 3.2.5. *In addition to the standing assumptions, suppose that H is discrete and μ_H is counting measure. Let $\mathscr{A} \subseteq L^2(G)$ be a countable family, and let*

$$J(\alpha) = \overline{\operatorname{span}}\{(Zf)(\alpha) : f \in \mathscr{A}\}$$

for a.e. $\alpha \in \hat{H}$. For constants $0 < A \leq B < \infty$, the following are equivalent:

(i) $E^H(\mathscr{A})$ is a Riesz basis for $S^H(\mathscr{A})$ with bounds A, B.

(ii) For a.e. $\alpha \in \hat{H}$, $\{(Zf)(\alpha) : f \in \mathscr{A}\}$ is a Riesz basis for $J(\alpha)$ with bounds A, B.

If G is abelian and

$$\tilde{J}(\omega H^*) = \overline{\operatorname{span}}\{(\mathcal{T}f)(\omega H^*) : f \in \mathscr{A}\}$$

for a.e. $\omega H^* \in \hat{G}/H^*$, the conditions above are equivalent to:

(iii) For a.e. $\omega H^* \in \hat{G}/H^*$, $\{(\mathcal{T}f)(\omega H^*) : f \in \mathscr{A}\}$ is a Riesz basis for $\tilde{J}(\omega H^*)$ with bounds A, B.

Proof. Recall that discrete abelian groups are dual to compact abelian groups, with counting measures dual to probability measures. Hence $\mu_{\hat{H}}(\hat{H}) = 1$. The theorem now follows from Theorem 2.1.3 in the same way that Theorem 3.2.4 followed from Theorem 2.1.10. □

Strictly speaking, the previous theorem holds even if H is not discrete. However, when $\hat{H}$ is not compact, condition (ii) can never occur. See Remark 2.1.4.

Remark 3.2.6. When $\mathscr{A}$ consists of a single function $f \in L^2(G)$, the conditions in the previous theorems simplify even further. Let $\Omega_f = \{\alpha \in \hat{H} : (Zf)(\alpha) \neq 0\}$. Then condition (ii) of Theorem 3.2.4 is equivalent to

(ii') For a.e. $\alpha \in \Omega_f$, $A \leq \|(Zf)(\alpha)\|^2 \leq B$.

When H is discrete, we can likewise replace condition (ii) of Theorem 3.2.5 with

(ii') For a.e. $\alpha \in \hat{H}$, $A \leq \|(Zf)(\alpha)\|^2 \leq B$.

Similar considerations apply for fiberization in the abelian setting.

We end this section with a pair of results on Gabor systems with critical sampling. We will assume that G is abelian. A closed subspace $M \subseteq L^2(G)$ is called (H, H^*)-*translation/modulation-invariant*, or (H, H^*)-TMI, if $L_\xi M_\kappa f \in M$ whenever $f \in M$, $\xi \in H$, and $\kappa \in H^*$. TMI spaces have usually been called "shift/modulation invariant", or SMI, in the discrete case. Following the examples of Bownik and Ross [13] and Jakobsen and Lemvig [45], we adopt the term TMI to emphasize that the subgroup involved need not be discrete.

Every family $\mathscr{A} \subseteq L^2(G)$ generates a *Gabor system* $\{L_\xi M_\kappa f : \xi \in H, \kappa \in H^*, f \in \mathscr{A}\}$. The closed linear span of this system is the smallest (H, H^*)-TMI space containing $\mathscr{A}$. The Zak transform has a long history of use for Gabor systems. We continue the tradition here. Our first result classifies (H, H^*)-TMI spaces in terms of the Zak transform. Our second result tells when Gabor systems are continuous frames.

The theorem below should be compared with Bownik [12, Theorem 5.1] and Cabrelli and Paternostro [15, Theorem 5.1]. Given a Borel subset $E \subseteq \hat{H} \times H\backslash G$, we denote

$$M_E = \{f \in L^2(G) : (\tilde{Z}f)(\alpha, Hx) = 0 \text{ for a.e. } (\alpha, Hx) \notin E\}.$$

Two Borel subsets of $\hat{H} \times H\backslash G$ are called *equivalent* if their symmetric difference has measure zero.

Theorem 3.2.7. *The (H, H^*)-TMI spaces in $L^2(G)$ are indexed by equivalence classes of Borel subsets of $\hat{H} \times H\backslash G$. A bijection maps $E \mapsto M_E$. For $\mathscr{A} \subseteq L^2(G)$, any countable dense subset $\mathscr{A}_0 \subseteq \mathscr{A}$ generates the same (H, H^*)-TMI space as does $\mathscr{A}$, and the corresponding subset of $\hat{H} \times H\backslash G$ is*

$$E = \{(\alpha, Hx) \in \hat{H} \times H\backslash G : (\tilde{Z}f)(\alpha, Hx) \neq 0 \text{ for some } f \in \mathscr{A}_0\}. \tag{3.21}$$

Proof. Since $H^* \cong (H\backslash G)^\wedge$, Pontryagin duality shows that $(\hat{H} \times H\backslash G)^\wedge \cong H \times H^*$. For $(\xi, \kappa) \in H \times H^*$, the corresponding character $X_{(\xi,\kappa)} \in (\hat{H} \times H\backslash G)^\wedge$ is given by $X_{(\xi,\kappa)}(\alpha, Hx) = \alpha(\xi)\kappa(x)$. By Lemma 3.2.2, the family $\mathcal{D} = (X_{(\xi,\kappa)})_{\xi \in H, \kappa \in H^*}$ is a Parseval determining set for $L^1(\hat{H} \times H\backslash G)$. Moreover, (3.10) shows that a subspace $M \subseteq L^2(G)$ is (H, H^*)-TMI if and only if $\tilde{Z}M \subseteq L^2(\hat{H} \times H\backslash G)$ is $\mathcal{D}$-MI. The proof follows from Proposition 2.1.2 once we observe that a range function

$J\colon \hat{H} \times H\backslash G \to \{\text{closed subsets of } \mathbb{C}\}$ identifies uniquely with the set

$$E = \{(\alpha, Hx) \in \hat{H} \times H\backslash G : J(\alpha, Hx) = \mathbb{C}\}.$$

Moreover, J is a measurable range function if and only if E is a Borel set. □

The next theorem generalizes a result of Arefijamaal [5, Theorem 2.6]. Also see Corollary 6.4.4 of Gröchenig [32], and the discussion that follows it.

Theorem 3.2.8. *Let $(\mathcal{M}, \mu_{\mathcal{M}})$ be a complete, σ-finite measure space, and let $\mathscr{A} = (f_t)_{t\in\mathcal{M}} \subseteq L^2(G)$ be a jointly measurable family of functions. Fix a countable dense subset $\mathscr{A}_0 \subseteq \mathscr{A}$, and let $E \subseteq \hat{H}\times H\backslash G$ be as in* (3.21). *For constants $0 < A \leq B < \infty$, the following are equivalent:*

(i) The Gabor system generated by $\mathscr{A}$ is a continuous frame for its closed linear span, with bounds A, B.

(ii) For a.e. $(\alpha, Hx) \in E$,

$$A \leq \int_{\mathcal{M}} |(\tilde{Z} f_t)(\alpha, Hx)|^2 \, d\mu_{\mathcal{M}}(t) \leq B.$$

Proof. As in the proof of Theorem 3.2.4, the family $\tilde{Z}\mathscr{A} = (\tilde{Z} f_t)_{t\in\mathcal{M}} \subseteq L^2(\hat{H}\times H\backslash G)$ is jointly measurable. The theorem now follows from Theorem 2.1.10 in the same way that Theorem 3.2.7 followed from Proposition 2.1.2. □

3.3. Dual integrable representations of LCA groups

We now turn our attention to a more general problem. Given a representation of a locally compact group on a Hilbert space $\mathcal{H}$, we would like to know when the orbit of a family of vectors $\mathscr{A} \subseteq \mathcal{H}$ makes a continuous frame in $\mathcal{H}$. We give an answer for a large class of representations of LCA groups.

Throughout this section, $\mathcal{G}$ will denote a fixed, second countable LCA group. Its Haar measure is $\mu_{\mathcal{G}}$, its dual group is $\hat{\mathcal{G}}$, and the dual Haar measure on $\hat{\mathcal{G}}$ is $\mu_{\hat{\mathcal{G}}}$. For $x \in \mathcal{G}$, the corresponding character of $\hat{\mathcal{G}}$ is X_x; that is, $X_x(\alpha) = \alpha(x)$. We set $\mathcal{D} = (X_x)_{x \in G}$. As explained in Remark 3.2.3, $\mathcal{D}$ is a Parseval determining set for $L^1(\hat{\mathcal{G}})$.

A (unitary) *representation* of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ is a strongly continuous group homomorphism $\pi \colon \mathcal{G} \to U(\mathcal{H})$ into the unitary group of $\mathcal{H}$. We call π *dual integrable* if there is a function

$$[\cdot, \cdot] \colon \mathcal{H} \times \mathcal{H} \to L^1(\hat{\mathcal{G}}),$$

called a *bracket* for π, such that

$$\langle \varphi, \pi(x)\psi \rangle = \int_{\hat{\mathcal{G}}} [\varphi, \psi](\alpha) \cdot \overline{\alpha(x)} \, d\mu_{\hat{\mathcal{G}}}(\alpha) \quad \text{for all } \varphi, \psi \in \mathcal{H} \text{ and } x \in \mathcal{G}.$$

In other words, a representation is dual integrable when all of its matrix elements lie in the image of the Fourier transform $L^1(\hat{\mathcal{G}}) \to C_0(\mathcal{G})$. The bracket gives the inverse Fourier transform of a matrix element. Consequently, the bracket is unique when it exists.

Dual integrable representations were introduced in the abstract setting by Hernández, Šikić, Weiss, and Wilson in [40]. Concrete versions of the bracket have been around much longer. Early uses appear in Jia and Michelli [46] and de Boor, DeVore, and Ron [22, 21]. An analog of dual integrable representations for possibly nonabelian countable discrete groups was recently developed by Barbieri, Hernández, and Parcet in [8]. Another version for square integrable functions over the Heisenberg group appears in Barbieri, Hernández, and Mayeli [7].

In this section, we fix a dual integrable representation π acting on a *separable* Hilbert space $\mathcal{H}$. Given $\varphi \in \mathcal{H}$, we denote

$$\langle \varphi \rangle = \overline{\operatorname{span}}\{\pi(x)\varphi : x \in \mathcal{G}\}.$$

We begin by recalling some basic properties of the bracket from [40].

Proposition 3.3.1. *The bracket is a sesquilinear Hermitian map* $[\cdot,\cdot]\colon \mathcal{H} \times \mathcal{H} \to L^1(\hat{G})$. *Moreover, for* $\varphi, \psi \in \mathcal{H}$ *and* $x \in G$, *the following hold:*

(i) $[\varphi, \varphi] \geq 0$ *a.e.*

(ii) $|[\varphi, \psi]| \leq [\varphi, \varphi]^{1/2}[\psi, \psi]^{1/2}$ *a.e.*

(iii) $\varphi \perp \langle \psi \rangle$ *if and only if* $[\varphi, \psi] = 0$ *a.e.*

(iv) $[\pi(x)\varphi, \psi] = X_x \cdot [\varphi, \psi] = [\varphi, \pi(x^{-1})\psi]$

Our strategy for understanding the translation action of an abelian subgroup in Section 3.2 was to apply an isometry that intertwined that action with modulation. We employ the same method here. Our isometry will be based on the following notion. The terminology is our own invention.

Definition 3.3.2. Let π be a representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$. A family of vectors $(\theta_i)_{i\in I} \subseteq \mathcal{H}$ is called *orthogonal generators* for π if $\mathcal{H} = \bigoplus_{i\in I}\langle \theta_i \rangle$.

Every representation admits a family of orthogonal generators, as a well-known consequence of Zorn's Lemma. Normally, the choice of generators is far from unique. For instance, any family of functions $\{f_i\}_{i\in I} \subseteq L^2(\mathbb{R})$ for which $\{\operatorname{supp} \hat{f}_i\}_{I\in I}$ forms a partition of $\mathbb{R}$, is an orthogonal generating family of the regular representation of $\mathbb{R}$. As this example demonstrates, orthogonal generators abound, and the cardinality of the indexing set I can change dramatically from family to family.

In an abstract sense, the lack of a canonical family of orthogonal generators might seem annoying, but in a practical sense, it is an advantage. In what follows, we analyze a dual integrable representation in terms of its bracket, a family $(\theta_i)_{i\in I}$ of orthogonal generators, and $l^2(I)$. The abundance of orthogonal generating families only makes this analysis more flexible.

For the remainder of the paper, we fix a family $(\theta_i)_{i\in I} \subseteq \mathcal{H}$ of orthogonal generators for π. For $i \in I$, we denote

$$\Omega_i = \{\alpha \in \hat{\mathcal{G}} : [\theta_i, \theta_i] \neq 0\}.$$

We also write $\delta_i \in l^2(I)$ for the standard basis element corresponding to $i \in I$.

The next proposition is Corollary (3.2) of [40]. The corollary after it was partially explained in the proof of [40, Corollary (3.4)].

Proposition 3.3.3. *Let $\psi \in \mathcal{H}$, and denote*

$$\Omega_\psi = \{\alpha \in \hat{\mathcal{G}} : [\psi, \psi](\alpha) \neq 0\}, \tag{3.22}$$

which is well defined up to a set of measure zero. The function $T_\psi \colon \langle\psi\rangle \to L^2(\hat{\mathcal{G}})$ given by

$$T_\psi(\varphi) = \mathbf{1}_{\Omega_\psi} \frac{[\varphi, \psi]}{[\psi, \psi]^{1/2}} \quad \textit{for } \varphi \in \langle\psi\rangle \subseteq \mathcal{H} \tag{3.23}$$

maps $\langle\psi\rangle$ unitarily onto $L^2(\Omega_\psi, \mu_{\hat{\mathcal{G}}})$.

Corollary 3.3.4. *The function* $T\colon \mathcal{H} \to L^2(\hat{\mathcal{G}}; l^2(I))$ *given by*

$$T(\varphi)(\alpha) = \left(\mathbf{1}_{\Omega_i}(\alpha) \cdot \frac{[\varphi, \theta_i](\alpha)}{([\theta_i, \theta_i](\alpha))^{1/2}}\right)_{i \in I} \quad \text{for } \varphi \in \mathcal{H} \text{ and } \alpha \in \hat{\mathcal{G}}$$

is a linear isometry satisfying

$$T(\pi(x)\varphi) = X_x \cdot T(\varphi) \quad \text{for all } \varphi \in \mathcal{H} \text{ and } x \in \mathcal{G}. \tag{3.24}$$

In particular, $T(\mathcal{H})$ *is a* $\mathcal{D}$*-MI space in* $L^2(\hat{\mathcal{G}}; l^2(I))$*. The range function* $J_0\colon \hat{\mathcal{G}} \to$ $\{$*closed subspaces of* $l^2(I)\}$ *given by*

$$J_0(\alpha) = \overline{\operatorname{span}}\{\mathbf{1}_{\Omega_i}(\alpha) \cdot \delta_i : i \in I\}$$

corresponds to $T(\mathcal{H})$*, in the sense of Proposition 2.1.2(ii).*

Proof. For each $i \in I$, let $P_i\colon \mathcal{H} \to \langle \theta_i \rangle$ be orthogonal projection, and let $T_i = T_{\theta_i}\colon \langle \theta_i \rangle \to L^2(\hat{\mathcal{G}})$ be the map from Proposition 3.3.3. Given $\varphi \in \mathcal{H}$ and $i \in I$, Proposition 3.3.1 implies that

$$[\varphi, \theta_i](\alpha) = [P_i\varphi, \theta_i](\alpha) + [(1 - P_i)\varphi, \theta_i](\alpha) = [P_i\varphi, \theta_i](\alpha).$$

Consequently,

$$T_i P_i \varphi = \mathbf{1}_{\Omega_i} \frac{[\varphi, \theta_i]}{([\theta_i, \theta_i])^{1/2}};$$

in other words,

$$T(\varphi)(\alpha) = ((T_i P_i \varphi)(\alpha))_{i \in I}.$$

By Proposition 3.3.3, T maps the spaces $\langle\theta_i\rangle \subseteq \mathcal{H}$ isometrically into orthogonal subspaces of $L^2(\hat{G}; l^2(I))$. Since $\mathcal{H} = \bigoplus_{i\in I}\langle\theta_i\rangle$, T is a linear isometry. Proposition 3.3.1(iv) gives (3.24).

For $i, j \in I$, Proposition 3.3.1 quickly implies that $[\theta_i, \theta_j] = \delta_{i,j} \cdot [\theta_i, \theta_i]$, where $\delta_{i,j}$ is the Kronecker-delta. Thus,

$$T(\theta_i)(\alpha) = ([\theta_i, \theta_i](\alpha))^{1/2} \cdot \delta_i. \tag{3.25}$$

Since $\mathcal{H} = \overline{\text{span}}\{\pi(x)\theta_i : x \in \mathcal{G}, i \in I\}$, we have, in the language of Section 2.1,

$$T(\mathcal{H}) = \overline{\text{span}}\{T(\pi(x)\theta_i) : x \in \mathcal{G}, i \in I\} = \overline{\text{span}}\{X_x \cdot T(\theta_i) : x \in \mathcal{G}, i \in I\}$$

$$= E_{\mathcal{D}}(\{T(\theta_i)\}_{i\in I}).$$

By Proposition 2.1.2(iii), the range function associated with $T(\mathcal{H})$ is

$$J_0(\alpha) = \overline{\text{span}}\{T(\theta_i)(\alpha) : i \in I\} = \overline{\text{span}}\{\mathbf{1}_{\Omega_i}(\alpha) \cdot \delta_i : i \in I\}. \qquad \square$$

A closed subspace $M \subseteq \mathcal{H}$ is called π-*invariant* if $\pi(x)\varphi \in M$ whenever $\varphi \in M$ and $x \in \mathcal{G}$. The restriction of each $\pi(x)$ to M gives the *subrepresentation* of π on M. The subrepresentation is also dual integrable, with the same bracket. The next theorem classifies π-invariant subspaces of $\mathcal{H}$ in terms of range functions.

Given a range function $J \colon \hat{\mathcal{G}} \to \{\text{closed subspaces of } l^2(I)\}$, we denote $P_J(\alpha) \colon l^2(I) \to J(\alpha)$ for the orthogonal projection associated to $\alpha \in \hat{\mathcal{G}}$. We also write

$$V_J = \{\varphi \in \mathcal{H} : (T\varphi)(\alpha) \in J(\alpha) \text{ for a.e. } \alpha \in \hat{\mathcal{G}}\}.$$

We call two range functions *equivalent* when they agree a.e. on $\hat{\mathcal{G}}$.

Given a family $\mathscr{A} \subseteq \mathcal{H}$, we write

$$E(\mathscr{A}) = \{\pi(x)\varphi : x \in \mathcal{G}, \varphi \in \mathscr{A}\}$$

for its orbit under π, and

$$S(\mathscr{A}) = \overline{\operatorname{span}}\{\pi(x)\varphi : x \in \mathcal{G}, \varphi \in \mathscr{A}\}$$

for the π-invariant space it generates.

Theorem 3.3.5. *Let J_0 be as in Corollary 3.3.4. The π-invariant subspaces of $\mathcal{H}$ are indexed by equivalence classes of measurable range functions $J\colon \hat{\mathcal{G}} \to \{$closed subspaces of $l^2(I)\}$ satisfying*

$$J(\alpha) \subseteq J_0(\alpha) \quad \textit{for a.e. } \alpha \in \hat{\mathcal{G}}. \tag{3.26}$$

A bijection maps $J \mapsto V_J$.

If $\mathscr{A} \subseteq \mathcal{H}$ has a countable dense subset $\mathscr{A}_0 \subseteq \mathscr{A}$, then the range function $J\colon \hat{\mathcal{G}} \to \{$closed subspaces of $l^2(I)\}$ given by

$$J(\alpha) = \overline{\operatorname{span}}\{(T\varphi)(\alpha) : \varphi \in \mathscr{A}_0\}. \tag{3.27}$$

satisfies

$$V_J = S(\mathscr{A}_0) = S(\mathscr{A}).$$

Proof. By Corollary 3.3.4, $E \mapsto T(E)$ is a bijection between closed π-invariant subspaces of $\mathcal{H}$ and $\mathcal{D}$-MI spaces contained in $T(\mathcal{H})$. Moreover, $E = V_J$ if and only if $T(E) = M_J$, in the language of Proposition 2.1.2. Obviously $M_J \subseteq T(\mathcal{H}) = M_{J_0}$ if and only if J satisfies (3.26), so the theorem is a consequence of Proposition 2.1.2 and Remark 3.2.3. □

Representations of LCA groups are uniquely determined by associated projection-valued measures on the dual group. For background, we refer the reader to Folland [26, Sections 1.4 and 4.4]. Hernández et al. [40, Corollary (2.5)] have given the projection-valued measure associated with a dual integrable representation, in terms of the bracket. The next proposition gives the projection-valued measure associated with an invariant subspace of a dual integrable representation, in terms of T.

Proposition 3.3.6. *Let $J\colon \hat{\mathcal{G}} \to \{$closed subspaces of $l^2(I)\}$ be a measurable range function satisfying* (3.26). *For each $E \subseteq \hat{\mathcal{G}}$, define a projection $P(E)$ on V_J by the formula*

$$T(P(E)\varphi) = \mathbf{1}_E \cdot T(\varphi).$$

Then P is a regular V_J-projection-valued measure on $\hat{\mathcal{G}}$, and the subrepresentation of π on V_J is given by

$$\pi(x) = \int_{\hat{\mathcal{G}}} \alpha(x)\, dP(\alpha).$$

Proof. For each $\varphi, \psi \in \mathcal{H}$, define a complex-valued measure $P_{\varphi,\psi}$ on $\hat{\mathcal{G}}$ with the formula

$$P_{\varphi,\psi}(E) = \langle P(E)\varphi, \psi\rangle = \langle \mathbf{1}_E \cdot T\varphi, T\psi\rangle = \int_{\hat{G}} \mathbf{1}_E(\alpha) \cdot \langle (T\varphi)(\alpha), (T\psi)(\alpha)\rangle\, d\mu_{\hat{\mathcal{G}}}(\alpha)$$

$$= \int_{\hat{G}} \mathbf{1}_E(\alpha) \cdot [\varphi, \psi](\alpha)\, d\mu_{\hat{\mathcal{G}}}(\alpha).$$

In other words, $dP_{\varphi,\psi} = [\varphi,\psi]\, d\mu_{\hat{\mathcal{G}}}$. By Corollary (2.5) of [40],

$$\langle \pi(x)\varphi, \psi\rangle = \int_{\hat{\mathcal{G}}} \alpha(x) \cdot [\varphi,\psi](\alpha)\, d\mu_{\hat{\mathcal{G}}}(\alpha) = \int_{\hat{\mathcal{G}}} \alpha(x)\, dP_{\varphi,\psi}(\alpha).$$

This completes the proof. □

We now give the main results of this section, reducing frame and Riesz basis conditions on the orbit of a family $\mathscr{A} \subseteq \mathcal{H}$ to pointwise conditions on the fibers $J(\alpha)$ from (3.27). In the special case of a discrete LCA group with a cyclic dual integrable representation, the next two theorems were given by Hernández et al. [40, Proposition (5.3) and Theorem (5.7)].

Theorem 3.3.7. *Let $(\mathcal{M}, \mu_{\mathcal{M}})$ be a complete, σ-finite measure space, and let $\mathscr{A} = (\varphi_t)_{t\in\mathcal{M}} \subseteq \mathcal{H}$ be a family of vectors such that, for each $i \in I$, the function*

$$(t,\alpha) \mapsto [\varphi_t, \theta_i](\alpha)$$

is measurable on $\mathcal{M} \times \hat{\mathcal{G}}$. Let $\mathscr{A}_0 \subseteq \mathscr{A}$ be a countable dense subset, and let J be as in (3.27)*. For constants $0 < A \leq B < \infty$, the following are equivalent:*

(i) $E(\mathscr{A})$ is a continuous frame for $S(\mathscr{A})$ with bounds A, B. That is,

$$A\,\|\psi\|^2 \leq \int_{\mathcal{M}} \int_{\mathcal{G}} |\langle \psi, \pi(x)\varphi_t\rangle|^2\, d\mu_{\mathcal{G}}(x)\, d\mu_{\mathcal{M}}(t) \leq B\,\|\psi\|^2$$

for all $\psi \in S(\mathscr{A})$.

(ii) For a.e. $\alpha \in \hat{\mathcal{G}}$, $\{P_J(\alpha)[T\varphi_t(\alpha)] : t \in \mathcal{M}\}$ is a continuous frame for $J(\alpha)$ with bounds A, B. In other words,

$$A \|v\|_{l^2(I)}^2 \leq \int_{\mathcal{M}} |\langle v, T\varphi_t(\alpha)\rangle_{l^2(I)}|^2 \, d\mu_{\mathcal{M}}(t) \leq B \|v\|_{l^2(I)}^2$$

for a.e. $\alpha \in \hat{\mathcal{G}}$ and all $v \in J(\alpha)$.

Proof. By Corollary 3.3.4, the linear isometry $T \colon \mathcal{H} \to L^2(\hat{\mathcal{G}}; l^2(I))$ maps $S(\mathscr{A})$ unitarily onto $S_{\mathcal{D}}(T\mathscr{A})$, sending $E(\mathscr{A})$ to $E_{\mathcal{D}}(T\mathscr{A})$. For each $i \in I$, the function

$$(t, \alpha) \mapsto \mathbf{1}_{\Omega_i}(\alpha) \cdot \frac{[\varphi_t, \theta_i](\alpha)}{([\theta_i, \theta_i](\alpha))^{1/2}}$$

is measurable on $\mathcal{M} \times \hat{\mathcal{G}}$. Therefore

$$(t, \alpha, i) \mapsto ([T\varphi_t](\alpha))_i$$

is measurable on $\mathcal{M} \times \hat{\mathcal{G}} \times I$. By Corollary 2.1.9, the family $T\mathscr{A} = (T\varphi_t)_{t \in \mathcal{M}} \subseteq L^2(\hat{\mathcal{G}}; l^2(I))$ is jointly measurable. The theorem now follows immediately from Theorem 2.1.10 and Remark 3.2.3. □

Theorem 3.3.8. *In addition to the standing assumptions, suppose that $\mathcal{G}$ is discrete. For a countable family $\mathscr{A} \subseteq \mathcal{H}$ and constants $0 < A \leq B < \infty$, the following are equivalent:*

(i) $E(\mathscr{A})$ forms a Riesz basis for $S(\mathscr{A})$ with bounds A, B.

(ii) For a.e. $\alpha \in \hat{\mathcal{G}}$, $\{T\varphi(\alpha) : \varphi \in \mathscr{A}\}$ forms a Riesz sequence in $l^2(I)$ with bounds A, B.

Proof. It follows from Theorem 2.1.3 in the same way that Theorem 3.3.7 followed from Theorem 2.1.10. □

For completeness, we mention the following combination of Lemma (2.8) and Proposition (5.1) in [40].

Proposition 3.3.9. *In addition to the standing assumptions, assume that $\mathcal{G}$ is discrete. For a family $\mathscr{A} = (\theta_i)_{i\in I} \subseteq \mathcal{H}$, $E(\mathscr{A})$ is an orthonormal sequence if and only if $[\theta_i, \theta_j] = \delta_{i,j}$ a.e.*

Remark 3.3.10. Given a single vector $\psi \in \mathcal{H}$, we can replace $\mathcal{H}$ with $\langle\psi\rangle$ and take $\{\psi\}$ for our family of orthogonal generators. Then T becomes the function $T_\psi \colon \langle\psi\rangle \to L^2(\hat{\mathcal{G}})$ from (3.23). The range function $J_0 \colon \hat{\mathcal{G}} \to \{\text{closed subspaces of } \mathbb{C}\}$ assigns $\mathbb{C}$ to every element of the set Ω_ψ from (3.22), and $\{0\}$ to every element of its complement. Taking $\mathscr{A} = \{\psi\}$ in Theorem 3.3.7, we see that the following are equivalent for constants $0 < A \leq B < \infty$:

(i) The orbit $(\pi(x)\psi)_{x\in\mathcal{G}}$ is a continuous frame for $\langle\psi\rangle$ with bounds A, B.

(ii) For a.e. $\alpha \in \Omega_\psi$, $A \leq [\psi,\psi](\alpha) \leq B$.

This generalizes Theorem (5.7) of [40] for continuous frames. A similar analysis recovers [40, Proposition (5.3)] from Theorem 3.3.8.

Example 3.3.11. Below are three prominent examples of dual integrable representations.

(i) If $\mathcal{H}_0$ is any separable Hilbert space, $\mathcal{G}$ acts on $L^2(\hat{\mathcal{G}}; \mathcal{H}_0)$ via the *modulation representation* $\hat{\lambda}$ given by

$$\hat{\lambda}(x)\varphi(\alpha) = \alpha(x) \cdot \varphi(\alpha).$$

This representation is dual integrable, and its bracket is given by the formula

$$[\varphi, \psi](\alpha) = \langle \varphi(\alpha), \psi(\alpha) \rangle.$$

(ii) Let G be a second countable locally compact group. Any closed abelian subgroup $H \subseteq G$ acts on $L^2(G)$ by left translation. This representation is dual integrable, and the Zak transform gives a formula for the bracket. Indeed, (3.15) says that for $f, g \in L^2(G)$ and $\alpha \in \hat{H}$,

$$[f, g](\alpha) = \langle (Zf)(\alpha^{-1}), (Zg)(\alpha^{-1}) \rangle_{L^2(H \backslash G)}.$$

When G is abelian, the bracket can also be expressed in terms of the fiberization map. For $f, g \in L^2(G)$ and $\omega \in \hat{G}$, (3.16) says that

$$[f, g](\omega|_H) = \langle (\mathcal{T} f)(\omega^{-1} H^*), (\mathcal{T} g)(\omega^{-1} H^*) \rangle_{L^2(H^*)}.$$

Theorems 3.2.1, 3.2.4, and 3.2.5 can be recovered from Theorems 3.3.5, 3.3.7, and 3.3.8, respectively.

(iii) Let G be a second countable LCA group with a closed subgroup H. Then $H \times H^*$ acts on $L^2(G)$ by translation and modulation. This representation is dual integrable, and the Zak transform gives a formula for the bracket, as follows. For any $f, g \in L^2(G)$, $\xi \in H$, and $\kappa \in H^*$, (3.10) produces

$$\langle f, L_\xi M_\kappa g \rangle_{L^2(G)} = \langle \tilde{Z} f, \tilde{Z} L_\xi M_\kappa g \rangle_{L^2(H \times H \backslash G)}$$

$$= \int_{\hat{H}} \int_{H \backslash G} (\tilde{Z} f)(\alpha, Hx) \overline{(\tilde{Z} g)(\alpha, Hx)} \cdot \alpha(\xi) \overline{\kappa(x)} \, d\mu_{H \backslash G} d\mu_{\hat{H}}(\alpha)$$

$$= \int_{\hat{H}} \int_{H\backslash G} (\tilde{Z}f)(\alpha^{-1}, Hx)\overline{(\tilde{Z}g)(\alpha^{-1}, Hx)} \cdot \overline{\alpha(\xi)\kappa(x)}\, d\mu_{H\backslash G} d\mu_{\hat{H}}(\alpha).$$

Since $H^* \cong (G/H)^\wedge$, Pontryagin Duality identifies $\widehat{H^*}$ with $H\backslash G$. For $Hx \in H\backslash G$, the corresponding character $X_{Hx} \in \widehat{H^*}$ is given by $X_{Hx}(\kappa) = \kappa(x)$. Thus,

$$[f, g](\alpha, X_{Hx}) = (\tilde{Z}f)(\alpha^{-1}, Hx)\overline{(\tilde{Z}g)(\alpha^{-1}, Hx)}.$$

Theorems 3.2.7 and 3.2.8 can be deduced from Theorems 3.3.5 and 3.3.7.

We end with several equivalent conditions for dual integrability, continuing the list begun in [40, Corollary 3.4]. The equivalence of (i) and (ii) below was essentially given there. From a philosophical perspective, the theorem below is the basis for our work on dual integrable representations, and the thread that connects Sections 2.1, 3.2, and 3.3. We remind the reader that representations σ and σ' of $\mathcal{G}$ acting on Hilbert spaces $\mathcal{H}_\sigma$ and $\mathcal{H}'_\sigma$, respectively, are called *unitarily equivalent* if there is a unitary $U\colon \mathcal{H}_\sigma \to \mathcal{H}'_\sigma$ such that $U\sigma(x) = \sigma'(x)U$ for all $x \in \mathcal{G}$.

Theorem 3.3.12. *For a representation σ of $\mathcal{G}$, the following are equivalent:*

(i) σ is dual integrable, and the space on which it acts is separable.

(ii) There is a separable Hilbert space $\mathcal{H}_0$ for which σ is unitarily equivalent to a subrepresentation of the modulation representation on $L^2(\mathcal{G}; \mathcal{H}_0)$.

(iii) There is a second countable locally compact group G containing $\mathcal{G}$ as a closed subgroup, and σ is unitarily equivalent to the left translation action of $\mathcal{G}$ on a $\mathcal{G}$-TI subspace of $L^2(G)$.

Proof. That (iii) implies (i) is the content of Example 3.3.11(ii). Corollary 3.3.4 says that (i) implies (ii). Suppose (ii) holds. Without loss of generality, we may assume that $\mathcal{H}_0 = l^2(K)$ for some countable set K. Give K the structure of a cyclic group, and let $G = \mathcal{G} \times K$. Let $\gamma\colon \mathcal{G}\backslash G \to G$ be the Borel section with fundamental domain $\gamma(\mathcal{G}\backslash G) = K \subseteq G$. Then the Zak transform is a unitary map $Z\colon L^2(G) \to L^2(\hat{\mathcal{G}}; l^2(K))$ intertwining the translation action of $\mathcal{G}$ on $L^2(G)$ with modulation on $L^2(\hat{\mathcal{G}}; l^2(K))$. Following the unitary equivalence in (ii) with Z^{-1} proves (iii). $\square$

3.4. Addendum: Group frames and shift-invariant spaces

This section is an addendum to the original article [42]. Here we explain that, roughly speaking, every group frame comes from the translation action on a shift-invariant space, as described for abelian groups in Section 3.2. With Theorem 3.3.12, this means in particular that every abelian group frame comes from a dual integrable representation.

Let Γ be a second countable locally compact group (not necessarily abelian), let $\pi\colon \Gamma \to U(\mathcal{H}_\pi)$ be a unitary representation on a separable Hilbert space, and let $\mathscr{A} = \{f_i\}_{i\in I} \subseteq \mathcal{H}_\pi$ be a countable family of vectors whose orbit

$$E(\mathscr{A}) = \{\pi(x) f_i : x \in \Gamma, i \in I\}$$

is a continuous frame for $\mathcal{H}_\pi$. In other words, there are constants A and B with $0 < A \leq B < \infty$ such that

$$A \left\| g \right\|^2 \leq \sum_{i\in I} \int_\Gamma \left| \langle g, \pi(x) f_i \rangle \right|^2 d\mu_\Gamma(x) \leq B \left\| g \right\|^2$$

for every $g \in \mathcal{H}_\pi$. The purpose of this section is to make a very simple observation about the representation π.

Theorem 3.4.1. *Up to unitary equivalence, π is a subrepresentation of the regular representation of Γ on $L^2(\Gamma; \ell^2(I))$. In other words, there is a linear isometry $T\colon \mathcal{H}_\pi \to L^2(\Gamma; \ell^2(I))$ such that*

$$(T\pi(x)g)(y) = (Tg)(x^{-1}y) \qquad (x, y \in \Gamma; g \in \mathcal{H}_\pi).$$

When Γ is either discrete or abelian, this implies that the representation π is dual integrable in the sense of [8, 9] or [40], respectively. Consequently, every frame generated by a countable discrete group is described in [8, 9], and every frame generated by a second countable LCA group is described in [40, 42].

The proof of Theorem 3.4.1 is almost trivial. We only need to adapt a few standard theorems from the theory of finite group frames to the continuous setting, and then make a one-line calculation.

Both of the following lemmas are adapted from [35, Chapter 8]. Let $S\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ be the frame operator for the frame $E(\mathscr{A})$. That is,

$$Sg = \sum_{i \in I} \int_\Gamma \langle g, \pi(x) f_i \rangle \pi(x) f_i \, d\mu_\Gamma(x) \qquad (g \in \mathcal{H}_\pi).$$

Here, the vector-valued integral should be interpreted in the weak sense. In other words, Sg is the unique vector in $\mathcal{H}$ defined weakly by the relation

$$\langle Sg, h \rangle = \int_\Gamma \langle g, \pi(x) f_i \rangle \langle \pi(x) f_i, h \rangle \, d\mu_\Gamma(x) \qquad (h \in \mathcal{H}_\pi).$$

Lemma 3.4.2. *The frame operator S lies in the commutant of π.*

Proof. This is a simple calculation. For any $g \in \mathcal{H}_\pi$ and $y \in \Gamma$, we have

$$S\pi(y)g = \sum_{i\in I} \int_\Gamma \langle \pi(y)g, \pi(x)f_i\rangle \pi(x)f_i \, d\mu_\Gamma(x) = \sum_{i\in I} \int_\Gamma \langle g, \pi(y^{-1}x)f_i\rangle \pi(x)f_i \, d\mu(x)$$

$$= \sum_{i\in I} \int_\Gamma \langle g, \pi(x)f_i\rangle \pi(yx)f_i \, d\mu_\Gamma(x) = \pi(y)Sg. \qquad \square$$

Lemma 3.4.3. *The canonical tight frame associated with $E(\mathscr{A})$ is another group frame. In other words, it has the form $E(\mathscr{A}')$ for some other countable family $\mathscr{A}' \subseteq \mathcal{H}_\pi$.*

Proof. The canonical tight frame is $\{S^{-1/2}\pi(x)f_i : x \in \Gamma, i \in I\}$. Since S lies in the commutant of π, we have

$$S^{-1/2}\pi(x) = \pi(x)S^{-1/2}$$

for every $x \in \Gamma$. Thus, the canonical tight frame is $\{\pi(x)f_i' : x \in G, i \in I\}$, where $f_i' = S^{-1/2}f_i$. $\square$

Proof of Theorem 3.4.1. Let

$$T \colon \mathcal{H}_\pi \to L^2(\gamma \times I)$$

be the analysis operator for $E(\mathscr{A})$. That is,

$$(Tg)(x,i) = \langle g, \pi(x)f_i\rangle \qquad (g \in \mathcal{H}_\pi; x \in \Gamma; i \in I).$$

By Lemma 3.4.3, we may assume without loss of generality that the frame is Parseval, and hence that T is an isometry. Now simply observe that

$$(T\pi(y)g)(x,i) = \langle \pi(y)g, \pi(x)f_i \rangle = \langle g, \pi(y^{-1}x)f_i \rangle = (Tg)(y^{-1}x, i)$$

for every $x, y \in \Gamma$, $g \in \mathcal{H}_\pi$, and $i \in I$. Following T with the canonical identification

$$L^2(\Gamma \times I) \cong L^2(\Gamma; \ell^2(I))$$

gives the desired isometry. $\square$

Another interpretation of Theorem 3.4.1 is that every group frame is given by translation. In the special case where Γ is compact, a version of the corollary below appeared in [43].

Corollary 3.4.4. *There is a second countable locally compact group G containing Γ as a closed subgroup, and a linear isometry*

$$T \colon \mathcal{H}_\pi \to L^2(G)$$

such that

$$T\pi(x) = L_x T$$

for every $x \in \Gamma$.

Proof. Give I the structure of a discrete abelian group, and let $G = \Gamma \times I$. As in the proof of Theorem 3.4.1, we can take T to be the analysis operator for the canonical tight frame. $\square$

Corollary 3.4.5. *Let G be any second countable locally compact group containing Γ as a closed subgroup of infinite index. Then there is a closed subspace $V \subseteq L^2(G)$ which is invariant under left translation by Γ, such that the action of Γ on V by left translation is unitarily equivalent to π.*

If $U\colon \mathcal{H}_\pi \to V$ is the unitary equivalence described above, then $U\pi(x)f_i = L_x U f_i$ for all $x \in \Gamma$ and all $i \in I$, so U gives a unitary equivalence between the group frame $\{\pi(x)f_i : x \in G, i \in I\}$ and the frame of translates $\{L_x(Uf_i) : x \in G, i \in I\}$.

This implies, for instance, that every group frame generated by $\mathbb{Z}^n$ is given by integer shifts in $L^2(\mathbb{R}^n)$, and has therefore been described by Bownik [11]. Likewise, descriptions of every possible frame generated by a second countable LCA group appear in any of [10, 13, 14, 42, 49].

Proof. We use follow the notation of Section 2.2. Assume for the moment that $L^2(\Gamma\backslash G)$ is infinite dimensional. Applying Corollary 2.2.7, we find there are unitaries

$$L^2(G) \cong L^2(\Gamma \times \Gamma\backslash G) \cong L^2(\Gamma; L^2(\Gamma\backslash G)) \cong L^2(\Gamma; \ell^2(\mathbb{Z}))$$

which all preserve left translation by Γ. Since I is countable, Theorem 3.4.1 says that π is unitarily equivalent to a subrepresentation of the regular representation on $L^2(\Gamma; \ell^2(\mathbb{Z}))$. Hence it is unitarily equivalent left translation by Γ on some invariant subspace of $L^2(G)$.

It remains to prove that $\dim L^2(\Gamma\backslash G) = \infty$. First, we claim that every nonempty open subset $U \subseteq \Gamma\backslash G$ has positive measure. Let $q\colon G \to \Gamma\backslash G$ be the quotient map. In G, nonempty open sets have positive measure (see, for instance, [26, Proposition 2.19]), so since μ_G is a regular measure, $q^{-1}(U)$ contains a compact subset K with

$0 < \mu_G(K) < \infty$. By (2.17),

$$0 < \mu_G(K) = \int_G \mathbf{1}_K(x)\, d\mu_G(x) = \int_{\Gamma\backslash G} \int_\Gamma \mathbf{1}_K(\xi\gamma(\Gamma x))\, d\mu_\Gamma(\xi)\, d\mu_{\Gamma\backslash G}(\Gamma x)$$

$$\leq \int_{\Gamma\backslash G} \int_\Gamma \mathbf{1}_{q^{-1}(U)}(\xi\gamma(\Gamma x))\, d\mu_\Gamma(\xi)\, d\mu_{\Gamma\backslash G}(\Gamma x)$$

$$= \int_{\Gamma\backslash G} \int_\Gamma \mathbf{1}_U(\Gamma x)\, d\mu_\Gamma(\xi)\, d\mu_{\Gamma\backslash G}(\Gamma x) = \mu_{\Gamma\backslash G}(U) \cdot \mu_\Gamma(\Gamma),$$

so $\mu_{\Gamma\backslash G}(U) > 0$, as claimed.

Since $\Gamma\backslash G$ is Hausdorff, any open set with at least two points contains two disjoint open subsets, each of which has positive measure. Hence, $\Gamma\backslash G$ either contains an infinite sequence of open sets $U_1 \supseteq U_2 \supseteq \cdots$ with the property that $0 < \mu_{\Gamma\backslash G}(U_n) < \infty$ and $\mu_{\Gamma\backslash G}(U_n \setminus U_{n+1}) > 0$ for each n, or $\Gamma\backslash G$ contains an open set which is a point. In the first case, the characteristic functions $\mathbf{1}_{U_1}, \mathbf{1}_{U_2}, \dots$ all belong to $L^2(\Gamma\backslash G)$ and satisfy $\mathbf{1}_{U_{n+1}} \notin \operatorname{span}\{\mathbf{1}_{U_1}, \dots, \mathbf{1}_{U_n}\}$ for all n, so $L^2(\Gamma\backslash G)$ is infinite dimensional. In the second case, at least one right coset of Γ is open in G. By applying right multiplication, we see that all the right cosets of Γ are open in G, so $\Gamma\backslash G$ is discrete. In particular, every point in $\Gamma\backslash G$ has positive measure. Since $\Gamma\backslash G$ is infinite by hypothesis, it follows again that $\dim L^2(\Gamma\backslash G) = \infty$. □

3.41. Conclusion

The results in this section were proved with very little effort. Their value lies not in the techniques they required, but in the philosophy they embody: if we want to learn more about group frames, then we should continue the extremely productive line of research on shift-invariant spaces. That philosophy will be fully exploited for representations of compact groups in Chapter IV.

CHAPTER IV

ACTIONS OF COMPACT GROUPS

This chapter has been accepted for publication as [43].

4.1. The Zak transform of a compact subgroup

In Sections 4.1 – 4.3, G is a second countable locally compact group (not necessarily abelian), and $K \subseteq G$ is a compact subgroup. Our main result is the existence of an operator-valued Zak transform on $L^2(G)$ that treats left translation by K in a manner similar to the Fourier transform on $L^2(K)$. This operator will form the basis for our classification of K-invariant subspaces of $L^2(G)$ in Section 4.2, and for our analysis of frames formed by K-translates in Section 4.3.

The reader may consult [26, 41] for background on compact groups and their representations. We record a few of the basics here. Throughout the paper, we normalize Haar measure on K so that $|K| = 1$. The left and right translates of $f\colon K \to \mathbb{C}$ by $\xi \in K$ are denoted $L_\xi f$ and $R_\xi f$, respectively. That is,

$$(L_\xi f)(\eta) = f(\xi^{-1}\eta), \quad (R_\xi f)(\eta) = f(\eta\xi) \qquad (\eta \in K).$$

We give $L^2(K)$ the usual convolution and involution, namely

$$(f * g)(\xi) = \int_K f(\eta)g(\eta^{-1}\xi)\, d\eta \qquad (f, g \in L^2(K);\ \xi \in K)$$

and

$$(f^*)(\xi) = \overline{f(\xi^{-1})} \qquad (f \in L^2(K),\ \xi \in K).$$

These operations make $L^2(K)$ a Banach $*$-algebra.

The dual object of K is $\hat{K}$; it has one representative of each equivalence class of irreducible unitary representations of K. Each $\pi \in \hat{K}$ acts on a finite dimensional space, which we denote $\mathcal{H}_\pi$. Its dimension is $d_\pi = \dim \mathcal{H}_\pi$. The *Fourier transform* of $f \in L^2(K)$ evaluated at $\pi \in \hat{K}$ is the operator

$$\hat{f}(\pi) = \int_K f(\xi)\pi(\xi^{-1})\, d\xi \in B(\mathcal{H}_\pi),$$

where the integral is to be interpreted in the weak sense. For our purposes, the utility of the Fourier transform lies in the formulae

$$(L_\xi f)\hat{\ }(\pi) = \hat{f}(\pi)\pi(\xi^{-1}), \quad (R_\xi f)\hat{\ }(\pi) = \pi(\xi)\hat{f}(\pi) \qquad (f \in L^2(K),\ \xi \in K,\ \pi \in \hat{K}) \tag{4.1}$$

and

$$(f^*)\hat{\ }(\pi) = \hat{f}(\pi)^*, \quad (f * g)\hat{\ }(\pi) = \hat{g}(\pi)\hat{f}(\pi) \qquad (f, g \in L^2(K);\ \pi \in \hat{K}). \tag{4.2}$$

If $B(\mathcal{H}_\pi)$ is treated as a Hilbert space with inner product $\langle A, B\rangle = d_\pi \langle A, B\rangle_{\mathcal{HS}} = d_\pi \operatorname{tr}(B^*A)$, the Fourier transform may be viewed as a unitary

$$\mathcal{F}\colon L^2(K) \to \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi), \qquad \mathcal{F}f = (\hat{f}(\pi))_{\pi \in \hat{K}}.$$

This is called *Plancherel's Theorem.* When an orthonormal basis $e_1^\pi, \ldots, e_{d_\pi}^\pi \in \mathcal{H}_\pi$ is chosen for each $\pi \in \hat{K}$, we define the *matrix elements* $\pi_{i,j} \in C(K)$ by

$$\pi_{i,j}(\xi) = \langle \pi(\xi)e_j^\pi, e_i^\pi \rangle \qquad (\pi \in \hat{K};\ \xi \in K;\ i, j = 1, \ldots, d_\pi).$$

In other words, the matrix for $\pi(\xi)$ with respect to the chosen basis is $(\pi_{i,j}(\xi))_{i,j=1}^{d_\pi}$. For $f \in L^2(K)$, the (i,j)-entry of the matrix for $\hat{f}(\pi)$ over this basis is

$$\hat{f}(\pi)_{i,j} = \int_K f(\xi)\overline{\pi_{i,j}(\xi)}\,d\xi \qquad (f \in L^2(K);\ \pi \in \hat{K};\ i,j = 1,\dots,d_\pi).$$

The *contragredient* to $\pi \in \hat{K}$ is the representation $\overline{\pi}$ on $\mathcal{H}_\pi$ with matrix elements

$$\overline{\pi}_{i,j}(\xi) = \overline{\pi_{i,j}(\xi)} \qquad (\xi \in K;\, i,j = 1,\dots,d_\pi).$$

The contragredient of an irreducible representation is also irreducible. The *Peter-Weyl Theorem* asserts that

$$\{\sqrt{d_\pi}\pi_{i,j} : \pi \in \hat{K},\, i,j = 1,\dots,d_\pi\}$$

is an orthonormal basis for $L^2(K)$. In particular,

$$\|f\|_{L^2(K)}^2 = \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} d_\pi |\hat{f}(\pi)_{i,j}|^2 = \sum_{\pi \in \hat{K}} d_\pi \left\| \hat{f}(\pi) \right\|_{\mathcal{HS}}^2 \qquad (f \in L^2(K)). \tag{4.3}$$

Let $K\backslash G$ be the quotient space of *right* cosets of K in G. A *cross section* of $K\backslash G$ in G is a map $\tau\colon K\backslash G \to G$ that selects a representative of each coset. In other words, $\tau(Kx) \in Kx$ for every $Kx \in K\backslash G$. By a classic result of Feldman and Greenleaf [25], there is a Borel cross section $\tau\colon K\backslash G \to G$ which maps compact subsets of $K\backslash G$ to sets with compact closure in G. Fix such a cross section, and let $T\colon K \times K\backslash G \to G$ be the bijection

$$T(\xi, Kx) = \xi \cdot \tau(Kx) \qquad (\xi \in K,\ Kx \in K\backslash G). \tag{4.4}$$

By [42, Theorem 3.6], $K\backslash G$ admits a unique regular Borel measure with respect to which T is a measure space isomorphism. We shall always have this measure in mind when we treat $K\backslash G$ as a measure space.

Given a function $f\colon G \to \mathbb{C}$ and a coset $Kx \in K\backslash G$, we will denote $f_{Kx}\colon K \to \mathbb{C}$ for the function given by

$$f_{Kx}(\xi) = f(\xi \cdot \tau(Kx)) \qquad (\xi \in K). \tag{4.5}$$

Intuitively, we are treating the coset Kx like a copy of K itself, with the chosen representative $\tau(Kx)$ taking the role of the identity element. In this sense, f_{Kx} is just the restriction of f to Kx. Obviously,

$$(L_\xi f)_{Kx} = L_\xi(f_{Kx}) \qquad (\xi \in K,\ Kx \in K\backslash G). \tag{4.6}$$

Theorem 4.1.1. *There is a unitary*

$$Z\colon L^2(G) \to \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi, L^2(K\backslash G; \mathcal{H}_\pi))$$

given by

$$[(Zf)(\pi)u](Kx) = (f_{Kx})\hat{}\,(\pi)u \qquad (f \in L^2(G),\ \pi \in \hat{K},\ u \in \mathcal{H}_\pi,\ Kx \in K\backslash G). \tag{4.7}$$

Here $B(\mathcal{H}_\pi, L^2(K\backslash G; \mathcal{H}_\pi))$ is treated as a Hilbert space with inner product $\langle A, B\rangle = d_\pi \operatorname{tr}(B^* A)$, and the direct sum is that of Hilbert spaces.

For $f \in L^2(G)$, $\xi \in K$, and $\pi \in \hat{K}$, the unitary Z satisfies

$$[Z(L_\xi f)](\pi) = (Zf)(\pi)\,\pi(\xi^{-1}). \tag{4.8}$$

We call Z the *Zak transform* for the pair (G, K).

Proof. The measure space isomorphism $T\colon K \times K\backslash G \to G$ induces a unitary $U\colon L^2(G) \to L^2(K \times K\backslash G)$, namely

$$(Uf)(\xi, Kx) = f(\xi \cdot \tau(Kx)) = f_{Kx}(\xi) \qquad (f \in L^2(G),\ \xi \in K,\ Kx \in K\backslash G).$$

Follow this with the canonical unitary $V\colon L^2(K \times K\backslash G) \to L^2(K) \otimes L^2(K\backslash G)$, and then apply

$$\mathcal{F}_K \otimes \mathrm{id}\colon L^2(K) \otimes L^2(K\backslash G) \to [\bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi)] \otimes L^2(K\backslash G).$$

Finally, make the natural identifications

$$[\bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi)] \otimes L^2(K\backslash G) \cong \bigoplus_{\pi \in \hat{K}} [B(\mathcal{H}_\pi) \otimes L^2(K\backslash G)] \cong \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes L^2(K\backslash G))$$

$$\cong \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi, L^2(K\backslash G; \mathcal{H}_\pi)).$$

The resulting composition is Z. The translation identity (4.8) follows directly from (4.6), (4.7), and the corresponding identity for the Fourier transform (4.1). $\square$

Remark 4.1.2. In the extreme case where K is all of G, the quotient $K\backslash G$ consists of a single point, and we can interpret $L^2(K\backslash G; \mathcal{H}_\pi)$ as simply being $\mathcal{H}_\pi$. Then the Zak transform reduces to the usual Fourier transform on $L^2(K)$, as long as the cross section τ chooses the identity element as the representative of the (single) coset of K in G.

In general, the choice of cross-section τ is noncanonical, and the operator Z depends on this choice. Nonetheless, Zak transforms associated with different cross-sections are easily related. Suppose that $\tau' \colon K\backslash G \to G$ is another cross-section with the required properties. For each $Kx \in K\backslash G$, there is an element $\eta_{Kx} \in K$ such that $\tau'(Kx) = \eta_{Kx}\tau(Kx)$. Denoting Z_τ and $Z_{\tau'}$ for the versions of the Zak transform obtained using τ and τ', respectively, we apply (4.7) and (4.1) to obtain the following relation for any $f \in L^2(G)$, $\pi \in \hat{K}$, $u \in \mathcal{H}_\pi$, and $Kx \in K\backslash G$:

$$[(Z_{\tau'}f)(\pi)u](Kx) = \pi(\eta_{Kx})[(Z_\tau f)(\pi)u](Kx).$$

In other words, $Z_{\tau'}$ can be obtained from Z_τ by applying post composition with $\pi(\eta_{Kx})$ at each point $Kx \in K\backslash G$ and in every coordinate $\pi \in \hat{K}$.

As with the usual Fourier transform on $L^2(K)$, there is another, basis-dependent version of the Zak transform that sometimes makes computation more convenient. When an orthonormal basis $e_1^\pi, \ldots, e_{d_\pi}^\pi$ is chosen for $\mathcal{H}_\pi$, the space $B(\mathcal{H}_\pi, L^2(K\backslash G; \mathcal{H}_\pi))$ can be identified with $M_{d_\pi}(L^2(K\backslash G))$ by mapping the operator A to the matrix whose (i,j)-entry is the function

$$Kx \mapsto \langle (Ae_j^\pi)(Kx), e_i^\pi \rangle \qquad (Kx \in K\backslash G).$$

Under this identification, the inner product on $M_{d_\pi}(L^2(K\backslash G))$ corresponding to the one in the definition of the Zak transform is given by

$$\langle M, N \rangle = d_\pi \sum_{i,j=1}^{d_\pi} \langle M_{i,j}, N_{i,j} \rangle \qquad (M, N \in M_{d_\pi}(L^2(K\backslash G))).$$

When this identification is made for each $\pi \in \hat{K}$, the Zak transform becomes a unitary

$$\tilde{Z}\colon L^2(G) \to \bigoplus_{\pi \in \hat{K}} M_{d_\pi}(L^2(K \backslash G)).$$

The translation formula (4.8) then becomes

$$[\tilde{Z}(L_\xi f)](\pi) = (\tilde{Z}f)(\pi) \cdot (\pi_{i,j}(\xi^{-1}))_{i,j=1}^{d_\pi} \qquad (f \in L^2(G),\ \xi \in K,\ \pi \in \hat{K}), \tag{4.9}$$

where the vector- and scalar-valued matrices multiply using the usual formula for matrix multiplication. For $f \in L^2(G)$ and $\pi \in \hat{K}$, the (i,j)-entry of $(\tilde{Z}f)(\pi)$ is the function in $L^2(K \backslash G)$ given by

$$Kx \mapsto \int_K f(\xi \tau(Kx)) \pi_{i,j}(\xi^{-1})\, d\xi \qquad (Kx \in K \backslash G). \tag{4.10}$$

For example, when K is a compact *abelian* group, each irreducible representation $\pi \in \hat{K}$ has dimension 1. Thus $M_{d_\pi}(L^2(K \backslash G))$ can be identified with $L^2(K \backslash G)$, and if we reinterpret the direct sum, we may view the Zak transform as a unitary

$$\tilde{\tilde{Z}}\colon L^2(G) \to \ell^2(\hat{K}; L^2(K \backslash G))$$

given by

$$[(\tilde{\tilde{Z}}f)(\alpha)](Kx) = \int_K f(\xi \tau(Kx)) \overline{\alpha(\xi)}\, d\xi \qquad (f \in L^2(G),\ \alpha \in \hat{K},\ Kx \in K \backslash G).$$

This agrees with the notion of Zak transform for an abelian subgroup described by the author in [42]. If G and K are both abelian, this definition is equivalent to the original notion of Zak transform as described by Weil in [63, p. 164–165]. That version of the Zak transform has a very long history in harmonic analysis. We refer the reader to [40] for a brief survey.

4.2. Range functions and translation invariance

A closed subspace $V \subseteq L^2(G)$ will be called *K-invariant* if $L_\xi f \in V$ whenever $f \in V$ and $\xi \in K$. In this section, we apply the Zak transform to classify the K-invariant subspaces of $L^2(G)$ in terms of *range functions*.

Definition 4.2.1. Let X be an indexing set, and let $\mathscr{H} = \{\mathcal{H}(x)\}_{x \in X}$ be a family of Hilbert spaces. A *range function* in $\mathscr{H}$ is a mapping

$$J \colon X \to \bigcup_{x \in X} \{\text{closed subspaces of } \mathcal{H}(x)\}$$

such that $J(x) \subseteq \mathcal{H}(x)$ for each $x \in X$. In other words, it is a choice of closed subspace $J(x) \subseteq \mathcal{H}(x)$ for each $x \in X$.

If J is a range function in $\{L^2(K \backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$, we define

$$V_J = \{f \in L^2(G) : \text{for all } \pi \in \hat{K}, \text{ the range of } (Zf)(\pi) \text{ is contained in } J(\pi)\}.$$

In terms of the Zak transform,

$$Z(V_J) = \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi, J(\pi)), \tag{4.11}$$

where we consider $B(\mathcal{H}_\pi, J(\pi))$ to be a closed subspace of $B(\mathcal{H}_\pi, L^2(K \backslash G; \mathcal{H}_\pi))$. The translation identity (4.8) for the Zak transform shows that V_J is K-invariant. Remarkably, every K-invariant subspace of $L^2(G)$ takes this form.

Theorem 4.2.2. *The mapping $J \mapsto V_J$ is a bijection between range functions in $\{L^2(K\backslash G;\mathcal{H}_\pi)\}_{\pi\in\hat{K}}$ and K-invariant subspaces of $L^2(G)$.*

A basis-dependent version of this theorem runs as follows. Choose orthonormal bases for each of the spaces $\mathcal{H}_\pi$, $\pi \in \hat{K}$, and let

$$\tilde{Z}\colon L^2(G) \to \bigoplus_{\pi\in\hat{K}} M_{d_\pi}(L^2(K\backslash G))$$

be the resulting basis-dependent Zak transform. For each $\pi \in \hat{K}$, we will think of the columns of $M_{d_\pi}(L^2(K\backslash G))$ as elements of $L^2(K\backslash G)^{\oplus d_\pi}$, the direct sum of d_π copies of $L^2(K\backslash G)$. Given a range function J in $\{L^2(K\backslash G)^{\oplus d_\pi}\}_{\pi\in\hat{K}}$, let

$$\tilde{V}_J = \{f \in L^2(G) : \text{for all } \pi \in \hat{K}, \text{ the columns of } (\tilde{Z}f)(\pi) \text{ lie in } J(\pi)\}.$$

Then $J \mapsto \tilde{V}_J$ is a bijection between range functions in $\{L^2(K\backslash G)^{\oplus d_\pi}\}_{\pi\in\hat{K}}$ and K-invariant subspaces of $L^2(G)$.

Range functions have a long history in the theory of translation invariance. Helson [37] and Srinivasan [57] seem to have the first results in this area. Their work was released at approximately the same time, and each cites the other, so it is not clear who deserves credit for this line of research. The idea of applying a Fourier-like transform and classifying invariant subspaces in terms of range functions has since been applied by a host of researchers in a variety of settings [1, 3, 4, 11, 12, 13, 14, 15, 19, 22, 38, 49]. Recently, the author [42] and Hernández, et al. [10] independently applied a version of the Zak transform to classify translation invariance by an abelian subgroup. The present theorem extends these results to the setting of compact groups.

We emphasize the novelty of applying this technique with a nonabelian subgroup. Of the results mentioned above, only Currey, et al. [19] treats a noncommutative case.[1] The theory of translation invariance in the nonabelian setting is only in its beginning stages. We hope that by first understanding the case of compact groups, where the representation theory is comparatively simple, we can help point a direction for understanding more general locally compact nonabelian groups.

The proof of Theorem 4.2.2 relies on a standard decomposition of actions of compact groups. If $\rho\colon K \to U(\mathcal{H}_\rho)$ is a unitary representation of K, then the *isotypical component* of $\pi \in \hat{K}$ in ρ is the invariant subspace $\mathcal{M}_\pi \subseteq \mathcal{H}_\rho$ spanned by all subspaces of $\mathcal{H}_\rho$ on which ρ is unitarily equivalent to π. Then

$$\mathcal{H}_\rho = \bigoplus_{\pi \in \hat{K}} \mathcal{M}_\pi. \tag{4.12}$$

Moreover, each $\mathcal{M}_\pi$ decomposes as a direct sum of irreducible subspaces on which ρ is equivalent to π. If $\mathrm{mult}(\pi, \rho)$ is the multiplicity of π in ρ, it follows that

$$\dim \mathcal{M}_\pi = d_\pi \cdot \mathrm{mult}(\pi, \rho) \qquad (\pi \in \hat{K}). \tag{4.13}$$

See [26, §5.1].

Given an invariant subspace $V \subseteq \mathcal{H}_\rho$, we will write ρ^V for the subrepresentation of ρ on V. The following can be deduced easily from [41, Theorem 27.44].

Lemma 4.2.3. *Let $\rho\colon K \to U(\mathcal{H}_\rho)$ be a unitary representation of K, with isotypical components $\mathcal{M}_\pi \subseteq \mathcal{H}_\rho$ for $\pi \in \hat{K}$.*

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[1] At least one other group of researchers has shown interest in generalizing the shift-invariance results of [11] to the compact nonabelian setting. An attempt at a classification theorem appears in [52].

(i) For each $\pi \in \hat{K}$, let $E_\pi \subseteq \mathcal{H}_\rho$ be the closed linear span of some invariant subspaces on which ρ is equivalent to π. If $\mathcal{H}_\rho = \bigoplus_{\pi \in \hat{K}} E_\pi$, then $E_\pi = \mathcal{M}_\pi$ for every $\pi \in \hat{K}$.

(ii) If $V \subseteq \mathcal{H}_\rho$ is an invariant subspace, then $V \cap \mathcal{M}_\pi$ is the isotypical component of $\pi \in \hat{K}$ in ρ^V.

The next lemma follows from Schur's Lemma and the Double Commutant Theorem for von Neumann algebras.

Lemma 4.2.4. *Let $\pi \in \hat{K}$. Then $B(\mathcal{H}_\pi) = \operatorname{span}\{\pi(\xi) : \xi \in K\}$.*

Proof of Theorem 4.2.2. Since Z is unitary, (4.11) shows that the mapping $J \mapsto V_J$ is injective. We need only prove that every K-invariant subspace $V \subseteq L^2(G)$ arises as such a V_J. To do this, we will first show that V decomposes as a direct sum of simpler pieces, and then we will leverage the Zak transform's translation property (4.8) on each piece.

Let ρ be the action of K on $L^2(G)$ by left translation. For each $\pi \in \hat{K}$, let

$$M_\pi = \{f \in L^2(G) : (Zf)(\sigma) = 0 \text{ for } \sigma \neq \pi\}.$$

We claim that M_π is the isotypical component of $\overline{\pi}$ in ρ. Fix an orthonormal basis $e_1^\pi, \dots, e_{d_\pi}^\pi \in \mathcal{H}_\pi$. For each $\pi \in \hat{K}$, $i = 1, \dots, d_\pi$, and nonzero $F \in L^2(K \backslash G; \mathcal{H}_\pi)$, we define $F_{\pi,i} \in L^2(G)$ by

$$(ZF_{\pi,i})(\sigma)e_j^\sigma = \begin{cases} d_\pi^{-1/2} \left\|F\right\|^{-1} \cdot F, & \text{if } \sigma = \pi \text{ and } i = j \\ 0, & \text{otherwise} \end{cases} \qquad (\sigma \in \hat{K};\ j = 1, \dots, d_\sigma)$$

Then $\langle F_{\pi,i}, F_{\pi,j}\rangle = \langle ZF_{\pi,i}, ZF_{\pi,}\rangle = \delta_{i,j}$, and one can check that

$$L_\xi F_{\pi,j} = \sum_{i=1}^{d_\pi} \overline{\pi}_{i,j}(\xi) \cdot F_{\pi,i}.$$

Hence $\mathrm{span}\{F_{\pi,i} : i = 1, \ldots, d_\pi\}$ is a K-invariant subspace of M_π on which ρ is equivalent to $\overline{\pi}$. Moreover,

$$M_\pi = \overline{\mathrm{span}}\{F_{\pi,i} : F \in L^2(K\backslash G; \mathcal{H}_\pi), i = 1, \ldots, d_\pi\}.$$

The claim follows from Lemma 4.2.3(i).

Now let $V \subseteq L^2(G)$ be a K-invariant subspace. By Lemma 4.2.3(ii),

$$V = \bigoplus_{\pi \in \hat{K}} V \cap M_\pi.$$

Since $(Zf)(\sigma) = 0$ for $f \in V \cap M_\pi$ and $\sigma \neq \pi$, we may view $W_\pi := Z(V \cap M_\pi)$ as a closed subspace of $B(\mathcal{H}_\pi, L^2(K\backslash G; \mathcal{H}_\pi))$. Let

$$J(\pi) = \overline{\mathrm{span}}\{Au : A \in W_\pi, u \in \mathcal{H}_\pi\} \subseteq L^2(K\backslash G; \mathcal{H}_\pi).$$

Clearly $W_\pi \subseteq B(\mathcal{H}_\pi, J(\pi))$. If we can upgrade this inclusion to equality, we will be able to conclude that

$$Z(V) = \bigoplus_{\pi \in \hat{K}} Z(V \cap M_\pi) = \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi, J(\pi)),$$

and the proof will be complete.

Fix any $A \in B(\mathcal{H}_\pi, J(\pi))$. We want to show that $A \in W_\pi$. A moment's thought shows that A is the sum of operators in $B(\mathcal{H}_\pi, J(\pi))$ whose kernels have codimension one. It is enough to show that each of those operators belongs to W_π. We may therefore assume that there is a unit norm vector $u \in \mathcal{H}_\pi$ such that $Av = 0$ for all $v \perp u$. Let $\epsilon > 0$ be arbitrary. Since $\operatorname{ran} A \subseteq J(\pi)$, we can find operators $B_1, \ldots, B_n \in W_\pi$ and nonzero vectors $v_1, \ldots, v_n \in \mathcal{H}_\pi$ such that

$$\left\| Au - \sum_{j=1}^{n} B_j v_j \right\|^2 < \epsilon.$$

We are going to produce an operator $B \in W_\pi$ with $Bu = \sum_{j=1}^n B_j v_j$ and $Bv = 0$ for $v \perp u$.

Here is the key step. Since $V \cap M_\pi$ is invariant under left translation by K, the identity (4.8) shows that $W_\pi = Z(V \cap M_\pi)$ is invariant under right multiplication by $\pi(\xi^{-1})$ for each $\xi \in K$. Therefore W_π is invariant under right multiplication by $B(\mathcal{H}_\pi) = \operatorname{span}\{\pi(\xi^{-1}) : \xi \in K\}$. In particular, we can precompose each $B_j \in W_\pi$ with another operator in $B(\mathcal{H}_\pi)$ to make $B_j' \in W_\pi$ satisfying $B_j' u = B_j v_j$ and $B_j' v = 0$ for all $v \perp u$. Then $B := B_1' + \cdots + B_n'$ belongs to W_π, and

$$\|A - B\|^2 = d_\pi \|Au - Bu\|^2 < d_\pi \epsilon.$$

Since W_π is closed and $\epsilon > 0$ was arbitrary, we conclude that $A \in W_\pi$. Therefore,

$$Z(V \cap M_\pi) = W_\pi = B(\mathcal{H}_\pi, J(\pi)),$$

as desired. $\square$

The preceding proof contained a fact that is useful in its own right.

Proposition 4.2.5. *Let J be a range function in $\{L^2(K \backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$, and let ρ_J be the representation of K on V_J given by left translation. Then the isotypical component* *of* $\pi \in \hat{K}$ *in* ρ_J *is*

$$\mathcal{M}_\pi = \{f \in V_J : (Zf)(\sigma) = 0 \text{ for } \sigma \neq \overline{\pi}\}.$$

In particular,

$$\operatorname{mult}(\pi, \rho_J) = \dim J(\overline{\pi}). \tag{4.14}$$

and

$$\dim V_J = \sum_{\pi \in \hat{K}} d_\pi \cdot \dim J(\overline{\pi}). \tag{4.15}$$

Proof. That $\mathcal{M}_\pi$ is the isotypical component of π in ρ_J was proven above. To see (4.14), simply observe that

$$\dim \mathcal{M}_\pi = \dim Z\mathcal{M}_\pi = d_\pi \cdot \dim J(\pi)$$

and apply (4.13). Then (4.15) follows from (4.12). □

Remark 4.2.6. K-invariant spaces are determined up to unitary equivalence by the dimensions of the spaces chosen by their range functions, in the following sense. Let J_1 and J_2 be two range functions in $\{L^2(K\backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$, and let V_1 and V_2 be the corresponding K-invariant subspaces of $L^2(G)$. Then there is a unitary map $U \colon V_1 \to V_2$ with the property that

$$UL_\xi = L_\xi U \qquad (\xi \in K)$$

if and only if

$$\dim J_1(\pi) = \dim J_2(\pi) \qquad (\pi \in \hat{K}).$$

This is a consequence of (4.14), since representations of compact groups are determined up to unitary equivalence by multiplicities of irreducible representations. Compare with Bownik's results on the dimension function for shift-invariant subspaces of $L^2(\mathbb{R}^n)$ [11, Theorem 4.10].

Theorem 4.2.7. *Let $\mathscr{A} \subseteq L^2(G)$ be an arbitrary family of functions, and let $S(\mathscr{A}) \subseteq L^2(G)$ be the K-invariant subspace generated by $\mathscr{A}$. That is,*

$$S(\mathscr{A}) = \overline{\operatorname{span}}\{L_\xi f : \xi \in K,\ f \in \mathscr{A}\}.$$

Then $S(\mathscr{A}) = V_J$, where

$$J(\pi) = \overline{\operatorname{span}}\{\operatorname{ran}(Zf)(\pi) : f \in \mathscr{A}\} \qquad (\pi \in \hat{K}).$$

Proof. If J and J' are two range functions in $\{L^2(K\backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$ with the property that $J(\pi) \subseteq J'(\pi)$ for all $\pi \in \hat{K}$, then it is easy to see that $V_J \subseteq V_{J'}$. Moreover, $V_{J'}$ contains $S(\mathscr{A})$ if and only if $J'(\pi)$ contains $\operatorname{ran}(Zf)(\pi)$ for all $f \in \mathscr{A}$, for every $\pi \in \hat{K}$. Since $S(\mathscr{A})$ is the smallest K-invariant space containing $\mathscr{A}$, the corresponding range function J must be such that $J(\pi)$ is the smallest closed subspace of $L^2(K\backslash G; \mathcal{H}_\pi)$ containing $\operatorname{ran}(Zf)(\pi)$ for all $f \in \mathscr{A}$, for every $\pi \in \hat{K}$. That subspace is precisely

$$\overline{\operatorname{span}}\{\operatorname{ran}(Zf)(\pi) : f \in \mathscr{A}\}. \qquad \square$$

Corollary 4.2.8. *$L^2(G)$ contains a function f with $\overline{\operatorname{span}}\{L_\xi f : \xi \in K\} = L^2(G)$ if and only if $G = K$.*

Proof. The K-invariant space $L^2(G)$ corresponds with the range function J' given by

$$J'(\pi) = L^2(K\backslash G; \mathcal{H}_\pi) \qquad (\pi \in \hat{K}).$$

If $K \subsetneq G$, then any $f \in L^2(G)$ has

$$\operatorname{rank}(Zf)(\pi) \leq d_\pi < \dim L^2(K\backslash G; \mathcal{H}_\pi) \qquad (\pi \in \hat{K}).$$

By the previous theorem, the range function J associated with $S(\{f\})$ has $J(\pi) = \operatorname{ran}(Zf)(\pi) \neq J'(\pi)$ for each $\pi \in \hat{K}$. Hence,

$$S(\{f\}) = V_J \neq V_{J'} = L^2(G).$$

When $G = K$, on the other hand, it is well known that every subrepresentation of the regular representation is cyclic. See, for instance, [31]. $\square$

We will now study the correspondence between range functions and K-invariant spaces in greater detail. Roughly speaking, we will see that the map $V_J \mapsto J$ allows us to view the lattice of K-invariant spaces as a much simpler lattice of linear subspaces. Many of the ideas that follow will appear again in our analysis of invariant subspaces of general representations of compact groups in Section 4.6.

To begin, we introduce the notion of direct sum for range functions. If J and J' are two range functions in the same family $\mathscr{H} = \{\mathcal{H}(x)\}_{x\in X}$, with the property that $J(x) \perp J'(x)$ for every $x \in X$, then we say that J and J' are *orthogonal*, and write $J \perp J'$. Given a family $\{J_\alpha\}_{\alpha\in A}$ of pairwise orthogonal range functions in $\mathscr{H}$, we denote $\oplus_{\alpha \in A} J_\alpha$ for the range function in $\mathscr{H}$ given by

$$[\bigoplus_{\alpha \in A} J_\alpha](x) = \bigoplus_{\alpha \in A} [J_\alpha(x)] \qquad (x \in X).$$

Let J and J' be two range functions in $\{L^2(K\backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$. For each $\pi \in \hat{K}$, we view $B(\mathcal{H}_\pi, J(\pi))$ and $B(\mathcal{H}_\pi, J'(\pi))$ as closed subspaces of $B(\mathcal{H}_\pi, L^2(K\backslash G; \mathcal{H}_\pi))$, with the inner product

$$\langle A, B \rangle = d_\pi \langle A, B \rangle_{\mathcal{HS}}.$$

Then $B(\mathcal{H}_\pi, J(\pi))$ is orthogonal to $B(\mathcal{H}_\pi, J'(\pi))$ if and only if $J(\pi) \perp J'(\pi)$. Since Z is unitary and

$$Z(V_J) = \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi, J(\pi)),$$

we conclude that

$$J \perp J' \iff V_J \perp V_{J'}. \tag{4.16}$$

Moreover, if $\{J_\alpha\}_{\alpha \in A}$ is a family of range functions in $\{L^2(K\backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$, then

$$J = \bigoplus_{\alpha \in A} J_\alpha \iff V_J = \bigoplus_{\alpha \in A} V_{J_\alpha}. \tag{4.17}$$

With these simple observations, we can easily describe all possible decompositions of V_J as a direct sum of irreducible subspaces.

Theorem 4.2.9. *Let J be a range function in $\{L^2(K\backslash G; \mathcal{H}_\pi)\}_{\pi \in \hat{K}}$. For each $\pi \in \hat{K}$, choose an orthonormal basis $\{F_i^\pi\}_{i \in I_\pi}$ for $J(\pi)$.*[2] *Then*

$$V_{\pi,i} := \{f \in L^2(G) : \operatorname{ran}(Zf)(\pi) \subseteq \operatorname{span}\{F_i^\pi\}, \text{ and } (Zf)(\sigma) = 0 \text{ for } \sigma \neq \pi \text{ in } \hat{K}\}$$

[2] If $J(\pi) = \{0\}$, we take I_π to be the empty set.

is an irreducible K-invariant space for each $\pi \in \hat{K}$ and $i \in I_\pi$, and

$$V_J = \bigoplus_{\pi \in \hat{K}} \bigoplus_{i \in I_\pi} V_{\pi,i}. \tag{4.18}$$

Moreover, every decomposition of V_J as a direct sum of irreducible K-invariant spaces occurs in this way.

In terms of the Zak transform, the direct sum decomposition (4.18) simply says that

$$Z(V_J) = \bigoplus_{\pi \in \hat{K}} \bigoplus_{i \in I_\pi} B(\mathcal{H}_\pi, \operatorname{span}\{F_i^\pi\}).$$

We can think of $\operatorname{span}\{F_i^\pi\}$ as being a copy of $\mathbb{C}$, so that $B(\mathcal{H}_\pi, \operatorname{span}\{F_i^\pi\})$ is like a copy of $\mathcal{H}_\pi^*$. It will therefore come as no surprise that the corresponding action of K on $B(\mathcal{H}_\pi, \operatorname{span}\{F_i^\pi\})$ is unitarily equivalent to $\overline{\pi}$.

Proof. For each $\pi \in \hat{K}$ and each $i = 1, \ldots, d_\pi$, let $J_{\pi,i}$ be the range function given by

$$J_{\pi,i}(\sigma) = \begin{cases} \operatorname{span}\{F_i^\pi\}, & \text{if } \sigma = \pi \\ \{0\}, & \text{if } \sigma \neq \pi \end{cases} \qquad (\sigma \in \hat{K}).$$

Then $V_{\pi,i} = V_{J_{\pi,i}}$, and the direct sum decomposition (4.18) follows immediately from (4.17). If $\rho_{\pi,i}$ is the action of K on $V_{\pi,i}$ by left translation, then $\rho_{\pi,i} \cong \overline{\pi}$ by (4.14). In particular, $V_{\pi,i}$ is irreducible.

Suppose

$$V_J = \bigoplus_{\alpha \in A} V_\alpha \tag{4.19}$$

is another decomposition of V_J into irreducible K-invariant spaces. Each V_α has the form V_{J_α} for some range function J_α, and (4.14) shows that $J_\alpha(\pi)$ is one dimensional for exactly one $\pi \in \hat{K}$, and trivial for all others. For that unique value of π, we choose a unit norm vector $G_\alpha \in J_\alpha(\pi)$.

Applying (4.17) again, we see that $J = \bigoplus_{\alpha \in A} J_\alpha$. In particular,

$$J(\pi) = \bigoplus_{\substack{\alpha \in A, \\ J_\alpha(\pi) \neq \{0\}}} J_\alpha(\pi) = \bigoplus_{\substack{\alpha \in A, \\ J_\alpha(\pi) \neq \{0\}}} \operatorname{span}\{G_\alpha\}$$

for each $\pi \in \hat{K}$. Hence $\{G_\alpha : \alpha \in A,\ J_\alpha(\pi) \neq \{0\}\}$ is an orthonormal basis for $J(\pi)$. Rearranging the decomposition (4.19) as

$$V_J = \bigoplus_{\pi \in \hat{K}} \bigoplus_{\substack{\alpha \in A, \\ J_\alpha(\pi) \neq \{0\}}} V_\alpha$$

shows it has the same form as (4.18). □

4.3. Frames of translates

There is a long tradition of combining range function classifications of invariant spaces with conditions for a family of translates to form a reproducing system. Bownik [11] seems to have the first results along these lines. His example was followed in [10, 13, 14, 19, 42, 49]. We now carry that tradition to the setting of compact, nonabelian subgroups. For our purposes, the relevant notion will be a continuous version of frames.

Definition 4.3.1. Let $\mathcal{H}$ be a separable Hilbert space, and let $(\mathcal{M}, \mu)$ be a σ-finite measure space. Let $\{f_x\}_{x \in \mathcal{M}}$ be an indexed family with the property that $x \mapsto \langle g, f_x \rangle$ is a measurable function on $\mathcal{M}$ for every $g \in \mathcal{H}$. Then $\{f_x\}_{x \in \mathcal{M}}$ is a *Bessel mapping* if there is a constant $B > 0$ such that

$$\int_{\mathcal{M}} |\langle g, f_x \rangle|^2 \, d\mu(x) \leq B \, \|g\|^2 \qquad \text{for every } g \in \mathcal{H}.$$

It is a *continuous frame* for $\mathcal{H}$ if there are constants $0 < A \leq B < \infty$ such that

$$A \, \|g\|^2 \leq \int_{\mathcal{M}} |\langle g, f_x \rangle|^2 \, d\mu(x) \leq B \, \|g\|^2 \qquad \text{for every } g \in \mathcal{H}.$$

The constants A and B are called *bounds*. If we can take $A = B$, the frame is *tight*. If we can take $A = B = 1$, it is a *Parseval frame*.

The reader unfamiliar with this notion may consult [2, 48], where it was originally developed. Further details are available in [27] and [53]. In the case where $\mathcal{M}$ is a discrete set equipped with counting measure, continuous frames reduce to the usual, discrete version. (The reader may even take this as a definition.) We will use the terms "frame" and "continuous frame" interchangeably.

The usual reproducing properties of discrete frames carry over to the continuous versions, with predictable modifications. Let $\{f_x\}_{x \in \mathcal{M}}$ be a Bessel mapping. The associated *analysis operator* $T \colon \mathcal{H} \to L^2(\mathcal{M})$ is defined by

$$(Tg)(x) = \langle g, f_x \rangle \qquad (g \in \mathcal{H}, \; x \in \mathcal{M});$$

its adjoint is the *synthesis operator* $T^* \colon L^2(\mathcal{M}) \to \mathcal{H}$,

$$T^* \phi = \int_{\mathcal{M}} \phi(x) f_x \, d\mu(x) \qquad (\phi \in L^2(\mathcal{M})),$$

where the vector-valued integral is interpreted in the weak sense. The *Gramian* is $\mathcal{G} = TT^*$, and the *frame operator* is $S = T^*T$. When our Bessel mapping is a continuous frame, the frame operator is positive and invertible, and $\{S^{-1/2} f_x\}_{x\in\mathcal{M}}$ is a continuous Parseval frame for $\mathcal{H}$, called the *canonical tight frame*. For Parseval frames, the frame operator is the identity map, and the Gramian is an orthogonal projection. Even when the frame is not tight, $\{S^{-1} f_x\}_{x\in\mathcal{M}}$ is another frame for $\mathcal{H}$ which satisfies

$$g = \int_{\mathcal{M}} \langle g, S^{-1} f_x \rangle f_x \, d\mu(x) \qquad (g \in \mathcal{H}).$$

Remark 4.3.2. The results in this paper apply for arbitrary second countable compact groups, which includes finite groups in particular. When K is finite, all of our results about continuous frames indexed by K can be interpreted in terms of discrete frames. We caution that it is necessary to reinterpret the frame bounds in this case, since Haar measure on K is normalized so that $|K| = 1$. In the special case where K is finite, a continuous frame over K having bounds A, B is the same as a discrete frame indexed by K having bounds $\mathrm{card}(K) \cdot A, \mathrm{card}(K) \cdot B$.

For a countable family $\mathscr{A} \subseteq L^2(G)$, we will denote

$$E(\mathscr{A}) = \{L_\xi f\}_{\xi\in K, f\in\mathscr{A}}$$

for the translates of $\mathscr{A}$. Recall that

$$S(\mathscr{A}) = \overline{\mathrm{span}}\, E(\mathscr{A})$$

is the K-invariant space generated by $\mathscr{A}$, and that $S(\mathscr{A}) = V_J$, with

$$J(\pi) = \overline{\operatorname{span}}\{\operatorname{ran}(Zf)(\pi) : f \in \mathscr{A}\} \qquad (\pi \in \hat{K}). \tag{4.20}$$

We would like to know under what circumstances $E(\mathscr{A})$ forms a continuous frame for $S(\mathscr{A})$. Our main result is as follows.

Theorem 4.3.3. *Let $\mathscr{A} \subseteq L^2(G)$ be a countable family of functions, and let J be the range function in* (4.20)*. For any constants $0 < A \leq B < \infty$ and any choice of orthonormal bases $e_1^\pi, \ldots, e_{d_\pi}^\pi \in \mathcal{H}_\pi$, $\pi \in \hat{K}$, the following are equivalent.*

(i) $E(\mathscr{A})$ is a continuous frame for $S(\mathscr{A})$ with bounds A, B. That is,

$$A\,\|g\|^2 \leq \sum_{f \in \mathscr{A}} \int_K |\langle g, L_\xi f\rangle|^2 \, d\xi \leq B\,\|g\|^2 \qquad (g \in S(\mathscr{A})). \tag{4.21}$$

(ii) For every $\pi \in \hat{K}$, $\{(Zf)(\pi)e_i^\pi : f \in \mathscr{A}, i = 1, \ldots, d_\pi\}$ is a discrete frame for $J(\pi)$ with bounds A, B.

This is in the spirit of [11, Theorem 2.3]. When K is compact and *abelian*, the theorem above reduces to [42, Theorem 5.4]. If G is also abelian, the same result was given in [10, Theorem 6.10]. Similar results appear in [10, 13, 14, 19, 42, 44, 49, 55].

The proof of Theorem 4.3.3 relies on the following lemma, which will also play a prominent role in Section 4.4. To each pair $f, g \in L^2(G)$, we associate the *matrix element* $V_f g \in C(K)$ given by

$$(V_f g)(\xi) = \langle g, L_\xi f\rangle \qquad (\xi \in K).$$

Lemma 4.3.4. *For $f, g \in L^2(G)$ and $\pi \in \hat{K}$,*

$$(V_f g)\hat{}\,(\pi) = (Zf)(\pi)^*(Zg)(\pi).$$

Proof. Fix an orthonormal basis $e_1^\pi, \ldots, e_{d_\pi}^\pi$ for each $\mathcal{H}_\pi$, $\pi \in \hat{K}$. For $f, g \in L^2(G)$, $\pi \in \hat{K}$, and $i, j = 1, \ldots, d_\pi$, the (i,j)-entry of the matrix for $(V_f g)\hat{}\,(\pi)$ with respect to this basis is

$$(V_f g)\hat{}\,(\pi)_{i,j} = \int_K \int_G g(x)\overline{(L_\xi f)(x)}\, dx\, \overline{\pi_{i,j}(\xi)}\, d\xi.$$

Applying the measure space isomorphism $G \to K\backslash G \times K$ from (4.4), we see this is equal to

$$\begin{aligned} &\int_K \int_{K\backslash G} \int_K g_{Kx}(\eta)\overline{f_{Kx}(\xi^{-1}\eta)}\, d\eta\, d(Kx)\, \overline{\pi_{i,j}(\xi)}\, d\xi \\ &= \int_K \int_{K\backslash G} (g_{Kx} * f_{Kx}^*)(\xi)\, d(Kx)\, \overline{\pi_{i,j}(\xi)}\, d\xi, \end{aligned}$$

where f_{Kx} and g_{Kx} are as defined in (4.5). We wish to reverse the order of integration above with Fubini's Theorem. Assuming for the moment that this is possible, we will have

$$(V_f g)\hat{}\,(\pi)_{i,j} = \int_K \int_{K\backslash G} (g_{Kx} * f_{Kx}^*)(\xi)\, d(Kx)\, \overline{\pi_{i,j}(\xi)}\, d\xi$$

$$= \int_{K\backslash G} \int_K (g_{Kx} * f_{Kx}^*)(\xi)\overline{\pi_{i,j}(\xi)}\, d\xi\, d(Kx) = \int_{K\backslash G} (g_{Kx} * f_{Kx}^*)\hat{}\,(\pi)_{i,j}\, d(Kx)$$

$$= \int_{K\backslash G} \langle (f_{Kx})\hat{}\,(\pi)^*(g_{Kx})\hat{}\,(\pi)e_j^\pi, e_i^\pi\rangle\, d(Kx) = \int_{K\backslash G} \langle (g_{Kx})\hat{}\,(\pi)e_j^\pi, (f_{Kx})\hat{}\,(\pi)e_i^\pi\rangle\, d(Kx)$$

$$= \int_{K\backslash G} \langle [(Zg)(\pi)e_j^\pi](Kx), [(Zf)(\pi)e_i^\pi](Kx)\rangle\, d(Kx) = \langle (Zg)(\pi)e_j^\pi, (Zf)(\pi)e_i^\pi\rangle$$

$$= [(Zf)(\pi)^*(Zg)(\pi)]_{i,j},$$

where we have applied the definition of the Zak transform (4.7) in the third to last equality. Once the above holds for all i and j, we will be able to conclude that

$$(V_f g)\hat{\ }(\pi) = (Zf)(\pi)^*(Zg)(\pi),$$

as desired.

It only remains to justify our use of Fubini's Theorem. To do so, we observe first that

$$|\pi_{i,j}(\xi)| = |\langle \pi(\xi)e_j^\pi, e_i^\pi\rangle| \leq \left\|\pi(\xi)e_j^\pi\right\| \|e_i^\pi\| = 1 \qquad (\xi \in K),$$

by Cauchy-Schwarz. Hence,

$$\int_{K\backslash G}\int_K |(g_{Kx} * f_{Kx}^*)(\xi)\pi_{i,j}(\xi)|\, d\xi\, d(Kx) \leq \int_{K\backslash G} \|g_{Kx} * f_{Kx}^*\|_{L^1(K)}\ d(Kx)$$

$$\leq \int_{K\backslash G} \|g_{Kx}\|_{L^1(K)} \|f_{Kx}\|_{L^1(K)}\ d(Kx)$$

$$\leq \left(\int_{K\backslash G} \|g_{Kx}\|_{L^1(K)}^2\ d(Kx)\right)^{1/2} \left(\int_{K\backslash G} \|f_{Kx}\|_{L^1(K)}^2\ d(Kx)\right)^{1/2}.$$

The proof will be finished if we can show that $\int_{K\backslash G} \|f_{Kx}\|_{L^1(K)}^2\, d(Kx) < \infty$ for all $f \in L^2(G)$. An application of Minkowski's Integral Inequality produces

$$\left(\int_{K\backslash G} \|f_{Kx}\|_{L^1(K)}^2\ d(Kx)\right)^{1/2} = \left(\int_{K\backslash G} \left|\int_K |f(\eta\tau(Kx))|\, d\eta\right|^2 d(Kx)\right)^{1/2}$$

$$\leq \int_K \left(\int_{K\backslash G} |f(\eta\tau(Kx))|^2\, d(Kx)\right)^{1/2} d\eta.$$

Let

$$E = \{\eta \in K : \int_{K\backslash G} |f(\eta\tau(Kx)|^2 \, d(Kx) < 1\}$$
$$F = \{\eta \in K : \int_{K\backslash G} |f(\eta\tau(Kx)|^2 \, d(Kx) \geq 1\}.$$

(These are well defined up to sets of measure zero.) Then

$$\left(\int_{K\backslash G} \|f_{Kx}\|^2_{L^1(K)} \, d(Kx) \right)^{1/2} \leq \int_K \left(\int_{K\backslash G} |f(\eta\tau(Kx))|^2 \, d(Kx) \right)^{1/2} d\eta$$

$$= \int_E \left(\int_{K\backslash G} |f(\eta\tau(Kx))|^2 \, d(Kx) \right)^{1/2} d\eta + \int_F \left(\int_{K\backslash G} |f(\eta\tau(Kx))|^2 \, d(Kx) \right)^{1/2} d\eta$$

$$\leq |E| + \int_F \int_{K\backslash G} |f(\eta\tau(Kx))|^2 \, d(Kx) \, d\eta \leq 1 + \int_K \int_{K\backslash G} |f(\eta\tau(Kx))|^2 \, d(Kx) \, d\eta$$

$$= 1 + \int_G |f(x)|^2 \, dx < \infty,$$

where we have once again applied the measure space isomorphism $K \times K\backslash G \to G$. This completes the proof. □

With this lemma in hand, Theorem 4.3.3 becomes an easy consequence of Plancherel's Theorem and our classification of K-invariant spaces.

Proof of Theorem 4.3.3. For any $f, g \in L^2(G)$, we use Plancherel's Theorem and Lemma 4.3.4 to perform the fundamental calculation

$$\int_K |\langle g, L_\xi f \rangle|^2 \, d\xi = \sum_{\pi \in \hat{K}} d_\pi \, \|(Zf)(\pi)^*(Zg)(\pi)\|^2_{\mathcal{HS}} \tag{4.22}$$

$$= \sum_{\pi \in \hat{K}} d_\pi \sum_{j=1}^{d_\pi} \sum_{i=1}^{d_\pi} |\langle (Zg)(\pi)e_j^\pi, (Zf)(\pi)e_i^\pi \rangle|^2.$$

On the other hand, the fact that Z is unitary implies

$$\|g\|^2 = \sum_{\pi\in\hat{K}} d_\pi \|(Zg)(\pi)\|^2_{\mathcal{HS}} = \sum_{\pi\in\hat{K}} d_\pi \sum_{j=1}^{d_\pi} \left\|(Zg)(\pi)e_j^\pi\right\|^2. \tag{4.23}$$

Suppose (i) holds. Fix $\pi \in \hat{K}$, and choose any $G \in J(\pi)$. Define $g \in L^2(G)$ by the formula

$$(Zg)(\sigma)e_j^\sigma = \begin{cases} d_\pi^{-1/2}G, & \text{if } \sigma = \pi \\ \\ 0, & \text{if } \sigma \neq \pi \end{cases} \qquad (\sigma \in \hat{K};\ j = 1, \ldots, d_\sigma).$$

Then $g \in V_J = S(\mathscr{A})$, by construction. It satisfies

$$\|g\|^2 = \|G\|^2,$$

by (4.23), and

$$\sum_{f\in\mathscr{A}} \int_K |\langle g, L_\xi f\rangle|^2 \, d\xi = \sum_{f\in\mathscr{A}} \sum_{i=1}^{d_\pi} |\langle G, (Zf)(\pi)e_i^\pi\rangle|^2,$$

by (4.22). Substituting these equations into (4.21) gives

$$A\|G\|^2 \leq \sum_{f\in\mathscr{A}} \sum_{i=1}^{d_\pi} |\langle G, (Zf)(\pi)e_i^\pi\rangle|^2 \leq B\|G\|^2.$$

In other words, (ii) holds.

Now assume (ii) is satisfied. For every $g \in S(\mathscr{A}) = V_J$ and every $\pi \in \hat{K}$, $(Zg)(\pi)e_j^\pi \in J(\pi)$. By (4.23) and the frame inequality,

$$A\|g\|^2 = \sum_{\pi \in \hat{K}} d_\pi \sum_{j=1}^{d_\pi} A \left\|(Zg)(\pi)e_j^\pi\right\|^2 \leq \sum_{\pi \in \hat{K}} d_\pi \sum_{j=1}^{d_\pi} \sum_{f \in \mathscr{A}} \sum_{i=1}^{d_\pi} |\langle (Zg)(\pi)e_j^\pi, (Zf)(\pi)e_i^\pi \rangle|^2.$$

Applying (4.22) to the last expression above, we see that

$$A\|g\|^2 \leq \sum_{f \in \mathscr{A}} \int_K |\langle g, L_\xi f\rangle|^2 \, d\xi.$$

A similar computation produces

$$\sum_{f \in \mathscr{A}} \int_K |\langle g, L_\xi f\rangle|^2 \, d\xi \leq B\|g\|^2.$$

This proves (i). □

4.4. Bracket analysis for compact group actions

We turn our attention now to a detailed study of group frames, as described in the introduction. In this section, we introduce a computational system known as a *bracket* for the analysis of representations of compact groups. Our primary motivation is the study of group frames with a single generator. We will see, however, that the bracket carries vital information about the structure of the representation itself, including its isotypical components and the multiplicities of irreducible representations. Several applications for the theory of group frames, including a complete classification of (compact) group frames with a single generator, appear in Section 4.5. Throughout, we fix a second countable compact group K, as in the previous sections, with Haar measure normalized so that $|K| = 1$. We also fix a unitary representation ρ of K, acting on a separable Hilbert space $\mathcal{H}_\rho$.

Our approach is motivated by the work of Weiss, et al. in [39]. Let $\mathcal{G}$ be a second countable locally compact abelian (LCA) group, with dual group $\hat{\mathcal{G}}$. Normalize Haar measures on $\mathcal{G}$ and $\hat{\mathcal{G}}$ so that the Plancherel theorem holds. A representation $\pi\colon \mathcal{G} \to U(\mathcal{H}_\pi)$ is called *dual integrable* if there is a *bracket*

$$[\cdot,\cdot]\colon \mathcal{H}_\pi \times \mathcal{H}_\pi \to L^1(\hat{\mathcal{G}})$$

such that

$$\langle f, \pi(x)g\rangle = \int_{\hat{\mathcal{G}}} [f,g](\alpha)\overline{\alpha(x)}\,d\alpha \qquad (f,g \in \mathcal{H}_\pi;\ x \in \mathcal{G}).$$

When $\mathcal{G}$ is identified with the dual of $\hat{\mathcal{G}}$ via Pontryagin Duality, this means that $\langle f, \pi(\cdot)g\rangle$ is the Fourier transform of $[f,g]$. The bracket provides an elegant description of frame properties for an orbit $\{\pi(x)f\}_{x\in\mathcal{G}}$.

Proposition 4.4.1 ([39, 42])**.** *For $f \in \mathcal{H}_\pi$ and constants A, B with $0 < A \le B < \infty$, the following are equivalent.*

(i) The orbit $\{\pi(x)f\}_{x\in\mathcal{G}}$ is a continuous frame for its closed linear span, with bounds A, B

(ii) For a.e. $\alpha \in \hat{\mathcal{G}}$, either $[f,f](\alpha) = 0$ or $A \le [f,f](\alpha) \le B$.

A possible difficulty with this approach is that, generally speaking, one may know a representation is dual integrable without being able to compute the bracket.[3] Suppose, however, that $\mathcal{G}$ is *compact* abelian. Then we can compute brackets as

[3]For certain kinds of representations, there are ways to recover the bracket even when $\mathcal{G}$ is not compact. Most of these methods involve variants of the Zak transform. See [39] and [42].

follows. Let π be *any* unitary representation of $\mathcal{G}$ on a separable Hilbert space $\mathcal{H}_\pi$. Then π decomposes as a direct sum of cyclic subrepresentations, each of which is unitarily equivalent to a subrepresentation of the regular representation. (See, for instance, [31].) By [39, Corollary 3.4], π is dual integrable. Let $[\cdot,\cdot]\colon \mathcal{H}_\pi \times \mathcal{H}_\pi \to L^1(\hat{\mathcal{G}})$ be a bracket for π. That is,

$$\langle f, \pi(x) g\rangle = [f,g]\hat{}\,(x) \qquad (f,g \in \mathcal{H};\ x \in \mathcal{G}).$$

Since $\mathcal{G}$ is compact, $[f,g]\hat{}$ lies in $C(\mathcal{G}) \subseteq L^1(\mathcal{G})$ for every $f,g \in \mathcal{H}_\pi$. Therefore we can apply Fourier inversion to recover the bracket from the matrix elements $\langle f, \pi(\cdot) g\rangle$:

$$[f,g](\alpha) = \langle f, \pi(\cdot) g\rangle\hat{}\,(\alpha^{-1}) \qquad (f,g \in \mathcal{H}_\pi;\ \alpha \in \hat{\mathcal{G}}).$$

These results suggest that, for our general compact group K with unitary representation ρ, it should be possible to analyze frames appearing as orbits of ρ using the (operator-valued) Fourier transform of the matrix elements

$$(V_g f)(\xi) := \langle f, \rho(\xi) g\rangle \qquad (f,g \in \mathcal{H}_\rho;\ \xi \in K).$$

This is indeed the case.

Definition 4.4.2. The *bracket* associated with ρ is the map

$$[\cdot,\cdot]\colon \mathcal{H}_\rho \times \mathcal{H}_\rho \to \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi)$$

given by

$$[f,g](\pi) = (V_g f)\hat{}\,(\pi) \qquad (\pi \in \hat{K}).$$

Here, as elsewhere, we consider $B(\mathcal{H}_\pi)$ to be a Hilbert space with inner product given by

$$\langle A, B\rangle = d_\pi \langle A, B\rangle_{\mathcal{HS}} = d_\pi \operatorname{tr}(B^* A).$$

Then $\bigoplus_{\pi\in\hat{K}} B(\mathcal{H}_\pi)$ is the Hilbert space direct sum.

Following the notation of [39], we will denote $\langle f\rangle \subseteq \mathcal{H}_\rho$ for the cyclic subspace generated by $f \in \mathcal{H}_\rho$. That is,

$$\langle f\rangle = \overline{\operatorname{span}}\{\rho(\xi)f : \xi \in K\} \qquad (f \in \mathcal{H}_\rho).$$

Our main result is the following.

Theorem 4.4.3. *For $f \in \mathcal{H}_\rho$ and constants A, B with $0 < A \leq B < \infty$, the following are equivalent.*

(i) The orbit $\{\rho(\xi)f\}_{\xi\in K}$ is a continuous frame for $\langle f\rangle$ with bounds A, B.

(ii) For every $\pi \in \hat{K}$, the nonzero eigenvalues of $[f, f](\pi)$ lie in the interval $[A, B]$.

When $\dim \mathcal{H}_\rho < \infty$, it is easy to tell when $\langle f\rangle = \mathcal{H}_\rho$ using the ranks of $[f, f](\pi)$, $\pi \in \hat{K}$; see Proposition 4.4.9 below. Thus, one can tell whether or not $\{\rho(\xi)f\}_{\xi\in K}$ is a frame for $\mathcal{H}_\rho$, and with what bounds, based solely on the eigenvalues of $[f, f](\pi)$, $\pi \in \hat{K}$, and their multiplicities. The condition that $\dim \mathcal{H}_\rho < \infty$ is always satisfied when $\{\rho(\xi)f\}_{\xi\in K}$ is a frame for $\mathcal{H}_\rho$; this is a consequence of Theorem 4.5.2, infra.

If Q_π denotes orthogonal projection of $\mathcal{H}_\pi$ onto $(\ker[f, f](\pi))^\perp$, then condition (ii) of the theorem above can be interpreted to say that $AQ_\pi \leq [f, f](\pi) \leq BQ_\pi$ for each $\pi \in \hat{K}$. (Compare with [8, Theorem A].) In the special case where K is compact *abelian*, Theorem 4.4.3 reduces to Proposition 4.4.1.

Tight frames generated by actions of *finite* nonabelian groups have been the focus of a flurry of recent activity [17, 18, 34, 58, 59, 60]. See [58, Theorem 6.18] and its generalization [60, Theorem 2.8] in particular for another characterization of *tight* frames that occur in this way. A nice summary of the state of the art circa 2013 appears in [61]; unfortunately the survey is already out of date, thanks in part to recent work by Waldron himself. This field is advancing rapidly.

Brackets have been used to analyze reproducing systems in $L^2(\mathbb{R}^n)$ since at least the work of Jia and Micchelli [46]. Weiss and his collaborators brought these techniques into the group-theoretic domain with [39], as described above. In the nonabelian setting, Hernández, et al. have developed notions of bracket maps for the Heisenberg group and for countable discrete groups [7, 8, 9].

The bracket defined above is related to the one that appears in [8, 9]. Suppose that K is finite (that is, both compact and discrete). Let us write $[\cdot,\cdot]_0 \colon \mathcal{H}_\rho \times \mathcal{H}_\rho \to B(L^2(K))$ for the bracket as developed in [8]. One can show that, for all $f, g \in \mathcal{H}_\rho$,

$$[f,g]_0(\phi) = \phi * V_g f \qquad (\phi \in L^2(K)).$$

Conjugating with the Fourier transform turns $[f,g]_0$ into left multiplication by $[f,g]$. One might say the papers [8, 9] study the convolution operator given by $V_g f$, where this paper studies its Fourier transform.

Much of our analysis relies on functions of positive type. We remind the reader that $\phi \in C(K)$ is said to be of *positive type* if

$$\int_K (f * f^*)(\xi)\, \phi(\xi)\, d\xi \geq 0 \qquad \text{for all } f \in L^1(K).$$

Equivalently, there is a unitary representation σ of K and a vector $f \in \mathcal{H}_\sigma$ such that

$$\phi(\xi) = \langle f, \sigma(\xi) f \rangle \qquad (\xi \in K).$$

The representation and the vector are unique in the following sense: If σ' is another representation of K with a cyclic vector $f' \in \mathcal{H}_{\sigma'}$ such that $\phi(\xi) = \langle f', \sigma'(\xi) f' \rangle$ for all $\xi \in K$, then there is a unitary $U \colon \mathcal{H}_{\sigma'} \to \mathcal{H}_\sigma$ intertwining σ' with σ and mapping $f' \mapsto f$. (See, for instance, [26, §3.3].) When σ is the regular representation and $f, g \in L^2(K)$, we have

$$\langle f, L_\xi g \rangle = \int_K f(\eta) g^*(\eta^{-1}\xi)\, d\eta = (f * g^*)(\xi) \qquad (\xi \in K). \tag{4.24}$$

For arbitrary $f \in L^2(K)$, this means that $\phi = f * f^*$ is a function of positive type. Up to unitary equivalence, the cyclic representations of K are precisely the subrepresentations of the regular representation ([31]); thus *every* function of positive type takes this form. In particular,

$$\phi^* = \phi, \tag{4.25}$$

and

$$\hat{\phi}(\pi) = (f * f^*)\hat{\ }(\pi) = \hat{f}(\pi)^* \hat{f}(\pi) \geq 0 \qquad (\pi \in \hat{K}). \tag{4.26}$$

(It is positive semidefinite.)

The bracket $[f, f]$ in Theorem 4.4.3 is the Fourier transform of the associated function of positive type

$$V_f f(\xi) = \langle f, \rho(\xi) f \rangle \qquad (\xi \in K).$$

Given $V_f f$, it is possible to reconstruct the Hilbert space $\langle f\rangle$, the restriction of ρ to $\langle f\rangle$, and the cyclic vector f. In other words, $V_f f$ contains complete information about the cyclic representation generated by f. Philosophically speaking, it must also be able to tell us when the orbit of f is a continuous frame for $\langle f\rangle$. Theorem 4.4.3 tells how to extract this information.

We will write $A^\dagger$ for the Moore-Penrose pseudoinverse of a bounded linear operator A. When A has closed range, $AA^\dagger$ is orthogonal projection onto the range of A, and $A^\dagger A$ is orthogonal projection onto $(\ker A)^\perp$.

Lemma 4.4.4. *For every $f \in \mathcal{H}_\rho$, there is a unique linear isometry $T_f\colon \langle f\rangle \to L^2(K)$ intertwining ρ with left translation, and sending f to a function of positive type. Explicitly,*

$$(T_f g)\hat{}\,(\pi) = ([f,f](\pi)^{1/2})^\dagger \cdot [g,f](\pi) \qquad (g \in \langle f\rangle,\ \pi \in \hat{K}). \tag{4.27}$$

Proof. Since the restriction of ρ to $\langle f\rangle$ is square integrable, the existence of a linear isometry $T_f\colon \langle f\rangle \to L^2(K)$ intertwining ρ with left translation and mapping f to a function of positive type is given by [23, Theorem 13.8.6]. Then $(T_f f)^* = T_f f$, and

$$(V_f f)(\xi) = \langle T_f f, L_\xi(T_f f)\rangle = [T_f f * (T_f f)^*](\xi) = (T_f f * T_f f)(\xi) \qquad (\xi \in K).$$

Since $(T_f f)\hat{}\,(\pi) \geq 0$ for all $\pi \in \hat{K}$, we conclude that

$$(T_f f)\hat{}\,(\pi) = [f,f](\pi)^{1/2} \qquad (\pi \in \hat{K}).$$

For any $g \in \langle f \rangle$, (4.24) gives

$$(V_f g)(\xi) = \langle T_f g, L_\xi T_f f \rangle = [(T_f g) * (T_f f)^*](\xi) = [(T_f g) * (T_f f)](\xi) \qquad (\xi \in K),$$

or equivalently,

$$[g, f](\pi) = (T_f f)\hat{}(\pi) \cdot (T_f g)\hat{}(\pi) \qquad (\pi \in \hat{K}). \tag{4.28}$$

Since $T_f g \in \langle T_f f \rangle$, Theorem 4.2.7 shows that

$$\operatorname{ran}(T_f g)\hat{}(\pi) \subseteq \operatorname{ran}(T_f f)\hat{}(\pi) = (\ker(T_f f)\hat{}(\pi))^\perp \qquad (\pi \in \hat{K}).$$

(Here we use the Fourier transform in place of the Zak transform; see Remark 4.1.2.) Applying $[(T_f f)\hat{}(\pi)]^\dagger = ([f, f](\pi)^{1/2})^\dagger$ to both sides of (4.28) establishes (4.27). In particular, T_f is uniquely determined. $\square$

Proposition 4.4.5. *The bracket has the following properties.*

(i) $[\cdot, \cdot]$ *is linear in the first variable, and conjugate linear in the second.*

(ii) For all $f, g \in \mathcal{H}_\rho$ *and* $\pi \in \hat{K}$*,*

$$[f, g](\pi) = [g, f](\pi)^*.$$

(iii) For all $f \in \mathcal{H}_\rho$ *and* $\pi \in \hat{K}$*,* $[f, f](\pi) \geq 0$*.*

(iv) For all $f, g \in \mathcal{H}_\rho$ *and* $A \in B(\mathcal{H}_\rho)$*,*

$$[Af, g] = [f, A^* g].$$

(v) For all $f, g \in \mathcal{H}_\rho$, $\pi \in \hat{K}$, and $\xi \in K$,

$$[f, \rho(\xi)g](\pi) = \pi(\xi) \cdot [f, g](\pi)$$

and

$$[\rho(\xi)f, g](\pi) = [f, g](\pi) \cdot \pi(\xi^{-1}).$$

(vi) For $f, g \in \mathcal{H}_\rho$, $f \perp \langle g \rangle$ if and only if $[f, g] = 0$.

More properties will be given in Propositions 4.4.7 and 4.4.8 below.

Proof. Item (i) follows from linearity of the Fourier transform and sesquilinearity of the map $(f, g) \mapsto V_g f$. To see (ii), apply (4.2) to the identity $V_f g = (V_g f)^*$. Equation (4.26) gives (iii), since $V_f f$ is a function of positive type. Apply the simple identity $V_g(Af) = V_{A^* g} f$ to get (iv). For (v), use (4.1) and the identities

$$V_{\rho(\xi)g} f = R_\xi(V_g f), \quad V_g(\rho(\xi)f) = L_\xi(V_g f) \qquad (f, g \in \mathcal{H}_\rho;\ \xi \in K).$$

For (vi), first assume that $f \perp \langle g \rangle$. Let P_g denote orthogonal projection of $\mathcal{H}_\rho$ onto $\langle g \rangle$, and apply (iv) to see that

$$[f, g] = [f, P_g g] = [P_g f, g] = 0.$$

Now suppose that $f, g \in \mathcal{H}_\rho$ satisfy $[f, g] = 0$. By Plancherel's Theorem, $V_g f = 0$. That is, $\langle f, \rho(\xi) g \rangle = 0$ for all $\xi \in K$. Hence $f \perp \langle g \rangle$. $\square$

When K is contained in a larger second countable, locally compact group G, the Zak transform provides a bracket for the action of K on $L^2(G)$ by left translation.

Indeed, Lemma 4.3.4 says precisely that

$$[f,g](\pi) = (Zg)(\pi)^*(Zf)(\pi) \qquad (f,g \in L^2(G);\ \pi \in \hat{K})$$

in this case. The theorem below shows that this example is universal; it is always possible to embed $\mathcal{H}_\rho$ as a K-invariant subspace of $L^2(G)$, for some larger group G containing K, in such a way that ρ becomes left translation by K.

Theorem 4.4.6. *There is a second countable, locally compact group G containing K as a closed subgroup, and a linear isometry $T\colon \mathcal{H}_\rho \to L^2(G)$ satisfying*

$$T\rho(\xi)f = L_\xi Tf \qquad (f \in \mathcal{H}_\rho,\ \xi \in K).$$

If Z is the Zak transform for the pair (G,K), then the bracket for ρ is given by

$$[f,g](\pi) = (ZTg)(\pi)^*(ZTf)(\pi) \qquad (f,g \in \mathcal{H}_\rho;\ \pi \in \hat{K}).$$

Proof. There is a countable family $\{f_i\}_{i\in I} \subseteq \mathcal{H}_\rho$ for which

$$\mathcal{H}_\rho = \bigoplus_{i\in I} \langle f_i \rangle.$$

For each $i \in I$, let $T_{f_i}\colon \langle f_i\rangle \to L^2(K)$ be the isometry from Lemma 4.4.4. Give I the structure of a discrete abelian group, and let $G = K \times I$. Given $g \in \mathcal{H}_\rho$, find the unique decomposition $g = \sum_{i\in I} g_i$ with $g_i \in \langle f_i\rangle$ for all i, and define

$$(Tg)(\xi,i) = (T_{f_i}g_i)(\xi) \qquad (\xi \in K,\ i \in I).$$

Then $T\colon \mathcal{H}_\rho \to L^2(G)$ is the desired isometry. $\square$

Proposition 4.4.7. *In addition to the properties listed in Proposition 4.4.5, the bracket satisfies the following.*

(i) For all $f, g \in \mathcal{H}_\rho$,

$$\langle f, g \rangle = \sum_{\pi \in \hat{K}} d_\pi \operatorname{tr}([f,g](\pi)). \tag{4.29}$$

(ii) For all $f, g \in \mathcal{H}_\rho$,

$$\|[f,g](\pi)\|_{\mathcal{HS}}^2 \leq \|[f,f](\pi)\|_{\mathcal{HS}} \, \|[g,g](\pi)\|_{\mathcal{HS}} \qquad (\pi \in \hat{K}). \tag{4.30}$$

(iii) If $f_n \to f$ in $\mathcal{H}_\rho$, then $[f_n, g] \to [f, g]$ for all $g \in \mathcal{H}_\rho$. In particular,

$$[f_n, g](\pi) \to [f, g](\pi)$$

for all $g \in \mathcal{H}_\rho$ and $\pi \in \hat{K}$.

Proof. By applying Theorem 4.4.6 if necessary, we may assume that $\mathcal{H}_\rho$ is a K-invariant subspace of $L^2(G)$ for some second countable locally compact group G containing K as a closed subgroup, that ρ is given by left translation of K, and that

$$[f,g](\pi) = (Zg)(\pi)^*(Zf)(\pi) \qquad (f, g \in \mathcal{H}_\rho;\ \pi \in \hat{K}),$$

where Z is the Zak transform for the pair (G, K). Now (iii) follows immediately from continuity of the Zak transform.

To prove (i), we simply compute

$$\langle f, g \rangle = \langle Zf, Zg \rangle = \sum_{\pi \in \hat{K}} d_\pi \langle (Zf)(\pi), (Zg)(\pi) \rangle_{\mathcal{HS}} = \sum_{\pi \in \hat{K}} d_\pi \operatorname{tr}([f,g](\pi)) \qquad (f, g \in \mathcal{H}_\rho).$$

For (ii), we use Cauchy-Schwarz for the Hilbert-Schmidt inner product to estimate

$$\|[f,g](\pi)\|_{\mathcal{HS}}^2 = \mathrm{tr}((Zf)(\pi)^*(Zg)(\pi)(Zg)(\pi)^*(Zf)(\pi))$$

$$= \mathrm{tr}((Zf)(\pi)(Zf)(\pi)^*(Zg)(\pi)(Zg)(\pi)^*)$$

$$= |\langle (Zg)(\pi)(Zg)(\pi)^*, (Zf)(\pi)(Zf)(\pi)^*\rangle_{\mathcal{HS}}|$$

$$\leq \|(Zg)(\pi)(Zg)(\pi)^*\|_{\mathcal{HS}} \|(Zf)(\pi)(Zf)(\pi)^*\|_{\mathcal{HS}}$$

$$= \|[g,g](\pi)\|_{\mathcal{HS}} \|[f,f](\pi)\|_{\mathcal{HS}}.$$

□

Equation (4.29) implies that vectors in $\mathcal{H}_\rho$ are uniquely determined by their bracket values. Specifically, if $f, g \in \mathcal{H}_\rho$ have $[f,h] = [g,h]$ for all $h \in \mathcal{H}_\rho$, then (4.29) shows that $\langle f, h\rangle = \langle g, h\rangle$, so that $f = g$. Propositions 4.4.5 and 4.4.7 together give the general feeling that the bracket behaves like a kind of operator-valued inner product on $\mathcal{H}_\rho$.[4] However, the bracket can tell us about much more than the linear and geometric properties of $\mathcal{H}_\rho$. It can tell us about ρ itself.

For each $\pi \in \hat{K}$, we will denote $\mathcal{M}_\pi$ for the isotypical component of π in ρ. In other words, $\mathcal{M}_\pi$ is the closed linear span of all invariant subspaces of $\mathcal{H}_\rho$ on which ρ is equivalent to π. We will write P_π for the orthogonal projection of $\mathcal{H}_\rho$ onto $\mathcal{M}_\pi$. Finally, when $V \subseteq \mathcal{H}_\rho$ is an invariant subspace, we denote ρ^V for the subrepresentation of ρ on V. Then we have the following proposition.

Proposition 4.4.8. *The bracket carries the following information about the isotypical components of ρ.*

---

[4]For representations of discrete groups, this idea was made more precise using the language of Hilbert modules and a slightly different notion of bracket in [9].

(i) For all $\pi \in \hat{K}$,

$$\mathcal{M}_\pi = \{f \in \mathcal{H}_\rho : [f,g](\sigma) = 0 \text{ for all } g \in \mathcal{H}_\rho \text{ and } \sigma \neq \overline{\pi}\}$$

$$= \{f \in \mathcal{H}_\rho : [f,f](\sigma) = 0 \text{ for } \sigma \neq \overline{\pi}\}.$$

(ii) For all $f, g \in \mathcal{H}_\rho$,

$$[f,g](\overline{\pi}) = [P_\pi f, g](\overline{\pi}) \qquad (\pi \in \hat{K}).$$

(iii) For all $f \in \mathcal{H}_\rho$

$$\operatorname{rank}[f,f](\pi) = \operatorname{mult}(\overline{\pi}, \rho^{\langle f \rangle}) \qquad (\pi \in \hat{K}).$$

In particular,

$$\dim \langle f \rangle = \sum_{\pi \in \hat{K}} d_\pi \cdot \operatorname{rank}[f,f](\pi).$$

Proof. As in the proof of the last proposition, we may assume that K is a closed subgroup of a second countable locally compact group G, that $\mathcal{H}_\rho$ is a K-invariant subspace of $L^2(G)$, and that ρ is left translation by K. If Z is the Zak transform for the pair (G,K), then the bracket is given by

$$[f,g](\pi) = (Zg)(\pi)^*(Zf)(\pi) \qquad (f,g \in \mathcal{H}_\rho;\ \pi \in \hat{K}).$$

For any $f \in \mathcal{H}_\rho$ and $\pi \in \hat{K}$, this implies in particular that $(Zf)(\pi) = 0$ if and only if $[f,f](\pi) = 0$. Moreover, the Cauchy-Schwarz type inequality (4.30) shows that $[f, f](\pi) = 0$ if and only if $[f, g](\pi) = 0$ for all $g \in \mathcal{H}_\rho$. Now (i) follows from Proposition 4.2.5.

For (ii), apply Proposition 4.2.5 to see that $(ZP_\pi f)(\overline{\pi}) = (Zf)(\overline{\pi})$.

Finally, (iii) follows from (4.14), Theorem 4.2.7, and the fact that

$$\operatorname{rank}[f, f](\pi) = \operatorname{rank}((Zf)(\pi)^*(Zf)(\pi)) = \operatorname{rank}((Zf)(\pi)) \qquad (\pi \in \hat{K}). \qquad \square$$

In many cases, statement (iii) above can be used to test whether a particular vector in $\mathcal{H}_\rho$ is cyclic for ρ.

Proposition 4.4.9. *Suppose that* $\operatorname{mult}(\pi, \rho) < \infty$ *for each* $\pi \in \hat{K}$*. Then* $f \in \mathcal{H}_\rho$ *is a cyclic vector for* ρ *if and only if*

$$\operatorname{rank}[f, f](\pi) = \operatorname{mult}(\overline{\pi}, \rho) \quad \textit{for every } \pi \in \hat{K}.$$

Moreover, when $\dim \mathcal{H}_\rho < \infty$*,* f *is a cyclic vector if and only if*

$$\sum_{\pi \in \hat{K}} d_\pi \cdot \operatorname{rank}[f, f](\pi) = \dim \mathcal{H}_\rho.$$

We can now prove our main result.

Proof of Theorem 4.4.3. By Lemma 4.4.4, we may assume that f is a function of positive type in $L^2(K)$, and that ρ is given by left translation. We are going to apply Theorem 4.3.3 with $G = K$ and $\mathscr{A} = \{f\}$. As explained in Remark 4.1.2, the Zak transform reduces to the Fourier transform in this case. In particular, Theorem 4.2.7 gives $\langle f \rangle = S(\mathscr{A}) = V_J$, where

$$J(\pi) = \operatorname{ran} \hat{f}(\pi) \qquad (\pi \in \hat{K}).$$

It remains to show that our condition (ii) is equivalent to condition (ii) in Theorem 4.3.3. For fixed $\pi \in \hat{K}$, we have $\hat{f}(\pi) \geq 0$, since f is a function of positive type. Choose an orthonormal basis $e_1^\pi, \ldots, e_{d_\pi}^\pi$ for $\mathcal{H}_\pi$ consisting of eigenvectors for $\hat{f}(\pi)$, with corresponding eigenvalues $\lambda_1^\pi \geq \ldots \geq \lambda_{d_\pi}^\pi \geq 0$. If $r_\pi = \operatorname{rank} \hat{f}(\pi)$, then the nonzero eigenvalues of $[f,f](\pi) = \hat{f}(\pi)^2$ are precisely $(\lambda_1^\pi)^2, \ldots, (\lambda_{r_\pi}^\pi)^2$. Now $\{\hat{f}(\pi) e_i^\pi\}_{i=1}^{d_\pi} = \{\lambda_i^\pi e_i^\pi\}_{i=1}^{d_\pi}$ is a discrete frame for $J(\pi) = \operatorname{span}\{e_1^\pi, \ldots, e_{r_\pi}^\pi\}$ with bounds A, B if and only if $A \leq (\lambda_1^\pi)^2, \ldots, (\lambda_{r_\pi}^\pi)^2 \leq B$. $\square$

Example 4.4.10. When ρ is irreducible, it is well known that any nonzero $f \in \mathcal{H}_\rho$ generates a continuous tight frame with bound $\|f\|^2 / (\dim \mathcal{H}_\rho)$. We can recover this fact as follows. First, Proposition 4.4.8(iii) shows that

$$\operatorname{rank}[f,f](\pi) = \begin{cases} 1, & \text{if } \pi = \overline{\rho} \\ 0, & \text{if } \pi \neq \overline{\rho} \end{cases} \qquad (\pi \in \hat{K}).$$

In particular, the operators $[f,f](\pi)$, $\pi \in \hat{K}$, have only one nonzero eigenvalue between them. Call that eigenvalue λ. By Theorem 4.4.3, $\{\rho(\xi) f\}_{\xi \in K}$ is a continuous tight frame with bound λ. Now use Proposition 4.4.7(i) to compute $\|f\|^2 = \lambda \cdot (\dim \mathcal{H}_\rho)$.

Example 4.4.11. Let $D_3 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order six. It has three irreducible representations: the trivial representation π_1, the one-dimensional representation π_2 given by $\pi_2(a) = 1$ and $\pi_2(b) = -1$, and the two-dimensional representation π_3 given by

$$\pi_3(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \pi_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the four-dimensional representation ρ given by

$$\rho(a) = \frac{1}{4}\begin{pmatrix} 1 & i\sqrt{3} & -3 & i\sqrt{3} \\ i\sqrt{3} & 1 & i\sqrt{3} & -3 \\ -3 & i\sqrt{3} & 1 & i\sqrt{3} \\ i\sqrt{3} & -3 & i\sqrt{3} & 1 \end{pmatrix} \quad \text{and} \quad \rho(b) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}.$$

Let $f = (3, 1, -1, 1)$. One can compute $[f, f](\pi_1) = 4$, $[f, f](\pi_2) = 4$, and

$$[f, f](\pi_3) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

By the dimension count in Proposition 4.4.9, $\langle f \rangle = \mathcal{H}_\rho = \mathbb{C}^4$. Applying Theorem 4.4.3, we see that the orbit of f forms a continuous frame for $\mathbb{C}^4$ with optimal bounds 2 and 4. When viewed as a discrete frame, the optimal bounds are 12 and 24. (See Remark 4.3.2.)

As this example demonstrates, bracket analysis can result in significant dimension reduction for the study of group frames. Suppose, for instance, that we want to know the optimal frame bounds for $\{\rho(\xi)f\}_{\xi \in K}$. A naive approach to this problem would be to compute the Gramian operator for the sequence $\{\rho(x)f\}_{x \in K}$ and find the range of its nonzero eigenvalues. In this example, that would mean computing the eigenvalues of a 6×6 matrix, which could be intractably difficult. Using bracket analysis, on the other hand, the largest matrix we had to analyze was 2×2.

4.5. Applications of bracket analysis

We now explore several applications of the bracket analysis developed in Section 4.4.

4.51. Block diagonalization of the Gramian

As we have just seen, the orbit $\{\rho(\xi)f\}_{\xi\in K}$ of a vector $f \in \mathcal{H}_\rho$ forms a frame only under special circumstances. However, compactness of K implies that it is always a *Bessel* mapping. Indeed, the Cauchy-Schwarz inequality produces

$$\int_K |\langle g, \rho(\xi)f\rangle|^2 \, d\xi \leq \int_K \|g\|^2 \cdot \|\rho(\xi)f\|^2 \, d\xi = \|f\|^2 \cdot \|g\|^2 \qquad (g \in \mathcal{H}).$$

In particular, the Gramian $\mathcal{G}\colon L^2(K) \to L^2(K)$ and the frame operator $S\colon \mathcal{H}_\rho \to \mathcal{H}_\rho$ are well-defined for any choice of $f \in \mathcal{H}_\rho$, whether or not $\{\rho(\xi)f\}_{\xi\in K}$ is a frame.

A direct computation shows the Gramian is given by

$$\mathcal{G}(\phi) = \phi * V_f f \qquad (\phi \in L^2(K)), \tag{4.31}$$

and the frame operator satisfies

$$V_h(Sg) = V_f g * V_h f \qquad (g, h \in \langle f \rangle).$$

Thus, S is defined uniquely by the relation

$$[Sg, h](\pi) = [f, h](\pi) \cdot [g, f](\pi) \qquad (g, h \in \langle f \rangle;\ \pi \in \hat{K}). \tag{4.32}$$

The Gramian and the frame operator are intimately connected through the linear isometry $T_f \colon \langle f \rangle \to L^2(K)$ from Lemma 4.4.4. Indeed, given any $g \in \langle f \rangle$, we compute

$$(T_f S g)\hat{}\,(\pi) = ([f,f](\pi)^{1/2})^\dagger \cdot [Sg,f](\pi) = ([f,f](\pi)^{1/2})^\dagger \cdot [f,f](\pi) \cdot [g,f](\pi)$$

$$= [f,f](\pi) \cdot ([f,f](\pi)^{1/2})^\dagger \cdot [g,f](\pi) = (\mathcal{G} T_f g)\hat{}\,(\pi) \qquad (\pi \in \hat{K}).$$

Therefore,

$$T_f S = \mathcal{G} T_f. \tag{4.33}$$

In fact, when $\{\rho(\xi) f\}_{\xi \in K}$ is a frame for $\langle f \rangle$, T_f is the analysis operator for the canonical tight frame. To see this, first observe that the range of T_f is $\langle T_f f \rangle$, the left ideal generated by $T_f f$. Let R be the operator on $\operatorname{ran} T_f$ given by

$$R(\phi) = \phi * (T_f f) \qquad (\phi \in \operatorname{ran} T_f).$$

For any $g \in \langle f \rangle$, we have

$$\langle g, \rho(\xi) f \rangle = \langle T_f g, L_\xi (T_f f) \rangle = [(T_f g) * (T_f f)^*](\xi) = [(T_f g) * (T_f f)](\xi)$$

$$= (R T_f g)(\xi) \qquad (\xi \in K).$$

In other words, the analysis operator $T \colon \langle f \rangle \to L^2(K)$ for the frame $\{\rho(\xi) f\}_{\xi \in K}$ is given by

$$T = R T_f.$$

Moreover, the computation above shows that $V_f f = (T_f f) * (T_f f)$, so

$$R^2 T_f = \mathcal{G} T_f = T_f S. \tag{4.34}$$

The operator R is positive semidefinite; for any $\phi \in \operatorname{ran} T_f$, we have

$$\langle \phi, R(\phi) \rangle = \langle \phi, \phi * (T_f f) \rangle = \langle \phi^* * \phi, T_f f \rangle = \int_K (\phi^* * \phi)(\xi) \cdot \overline{(T_f f)(\xi)}\, d\xi \geq 0,$$

since $\overline{T_f f}$ is also a function of positive type. Since T_f is a linear isometry, it follows from (4.34) that $T_f S^{1/2} = R T_f = T$. Equivalently, $T_f = T S^{-1/2}$, as desired.

One is often interested in the spectrum $\sigma(\mathcal{G})$ of the Gramian, since the optimal frame bounds are precisely the infimum and supremum of $\sigma(\mathcal{G}) \setminus \{0\}$. For a general positive semidefinite operator, finding the spectrum means diagonalization, which may be extremely difficult. For group frames, however, the realization of $\mathcal{G}$ as a convolution operator in (4.31) can take us a long way in this direction, as in the proposition below.

Proposition 4.5.1. *Fix any $f \in \mathcal{H}_\rho$, and let $\mathcal{G} \colon L^2(K) \to L^2(K)$ be the Gramian for the Bessel mapping $\{\rho(\xi) f\}_{\xi \in K}$. For each $\pi \in \hat{K}$, choose an orthonormal basis for $B(\mathcal{H}_\pi)$ with respect to the inner product $\langle A, B \rangle = d_\pi \operatorname{tr}(B^* A)$. Let $M_{[f,f](\pi)} \in M_{d_\pi^2}(\mathbb{C})$ be the matrix over this basis for the operator $M_{[f,f](\pi)} \colon B(\mathcal{H}_\pi) \to B(\mathcal{H}_\pi)$ given by*

$$M_{[f,f](\pi)}(A) = [f,f](\pi) \cdot A.$$

If $\hat{K} = \{\pi_1, \pi_2, \dots\}$, then $\mathcal{G}$ is unitarily equivalent to the block diagonal matrix

$$\tilde{\mathcal{G}} = \begin{pmatrix} M_{[f,f](\pi_1)} & & 0 \\ & M_{[f,f](\pi_2)} & \\ 0 & & \ddots \end{pmatrix},$$

and the Fourier transform $\mathcal{F}\colon L^2(K) \to \bigoplus_{\pi\in\hat{K}} B(\mathcal{H}_\pi)$ is a conjugating unitary. That is, $\tilde{\mathcal{G}} = \mathcal{F}\mathcal{G}\mathcal{F}^{-1}$.

Proof. This is obvious from the formulae (4.31), which gives the Gramian as a convolution operator, and (4.2), which says the Fourier transform turns convolution operators into multiplication operators. $\square$

Proposition 4.5.1 leads to an alternative proof of Theorem 4.4.3. Briefly: the spectrum of $\mathcal{G}$ is the union of the eigenvalues for $M_{[f,f](\pi)}$ as π runs through $\hat{K}$, and the eigenvalues for $M_{[f,f](\pi)}$ are the same as those for $[f,f](\pi)$. Now use the fact that a Bessel mapping is a frame if and only if the nonzero elements of $\sigma(\mathcal{G})$ are bounded away from zero, with the optimal frame bounds equal to the infimum and supremum of $\sigma(\mathcal{G}) \setminus \{0\}$, respectively.

4.52. Classification of K-frames

Continuous frames of the form $\{\rho(\xi)f\}_{\xi\in K}$ are sometimes called K-frames. We will say that ρ *admits* a K-frame if $\mathcal{H}_\rho$ has a continuous frame of this form. In that case, the orbit of f spans $\mathcal{H}_\rho$, so in particular ρ is cyclic. Greenleaf and Moskowitz [31, Theorem 1.10] have reduced the property of being cyclic to a count of multiplicities of irreducible representations. Explicitly, they have shown that ρ is cyclic if and only if $\mathrm{mult}(\pi,\rho) \leq d_\pi$ for each $\pi \in \hat{K}$. The following theorem refines this result for K-frames.

Theorem 4.5.2. *The following are equivalent.*

(i) ρ admits a K-frame.

(ii) ρ admits a Parseval K-frame.

(iii) ρ *is cyclic, and* $\dim \mathcal{H}_\rho < \infty$.

(iv) For all $\pi \in \hat{K}$, $\mathrm{mult}(\pi, \rho) \leq d_\pi$. *Moreover,* $\mathrm{mult}(\pi, \rho) = 0$ *for all but finitely many* $\pi \in \hat{K}$.

The result of Greenleaf and Moskowitz mentioned above says, in part, that every subrepresentation of the regular representation of K on $L^2(K)$ admits a cyclic vector. Theorem 4.5.2 shows that this result can not be improved using the language of frames. In particular, the regular representation admits a K-frame if and only if K is finite.

Proof. The equivalence of (iii) and (iv) is obvious from [31, Theorem 1.10] and the formula

$$\dim \mathcal{H}_\rho = \sum_{\pi \in \hat{K}} d_\pi \cdot \mathrm{mult}(\pi, \rho).$$

It remains to prove that (i), (ii), and (iv) are equivalent.

(i) $\implies$ (iv). Let $f \in \mathcal{H}_\rho$ be such that $\{\rho(\xi) f\}_{\xi \in K}$ is a continuous frame for $\mathcal{H}_\rho$, with lower frame bound $A > 0$. Since ρ is cyclic, [31, Theorem 1.10] shows that $\mathrm{mult}(\pi, \rho) \leq d_\pi$ for all $\pi \in \hat{K}$. By Proposition 4.4.7(i), Theorem 4.4.3, and Proposition 4.4.8(iii),

$$\|f\|^2 = \sum_{\pi \in \hat{K}} d_\pi \operatorname{tr}([f,f](\pi)) \geq \sum_{\pi \in \hat{K}} d_\pi A \cdot \operatorname{rank}([f,f](\pi)) = A \sum_{\pi \in \hat{K}} d_\pi \,\mathrm{mult}(\overline{\pi}, \rho).$$

Consequently, $\mathrm{mult}(\pi, \rho) = 0$ for all except finitely many $\pi \in \hat{K}$.

(iv) $\implies$ (ii). We are going to embed $\mathcal{H}_\rho$ as a translation-invariant subspace of $L^2(K)$. Recalling that the Zak transform for the pair (K, K) is the usual Fourier transform on $L^2(K)$ (see Remark 4.1.2), we can then use the results of Section 4.2 to analyze $\mathcal{H}_\rho$.

For each $\pi \in \hat{K}$, choose a subspace $J(\pi) \subseteq \mathcal{H}_\pi$ of dimension equal to $\operatorname{mult}(\overline{\pi}, \rho)$. Let

$$V_J = \{f \in L^2(K) : \operatorname{ran} \hat{f}(\pi) \subseteq J(\pi) \text{ for each } \pi \in \hat{K}\}$$

be the translation invariant subspace of $L^2(K)$ corresponding to the range function J. Since representations of K are determined up to unitary equivalence by multiplicities of irreducible representations, we may assume by (4.14) that $\mathcal{H}_\rho = V_J$, and that ρ is given by left translation. For each $\pi \in \hat{K}$, let $P_\pi \in B(\mathcal{H}_\pi)$ be orthogonal projection onto $J(\pi)$. Then

$$\sum_{\pi \in \hat{K}} d_\pi \, \|P_\pi\|_{\mathcal{HS}}^2 = \sum_{\pi \in \hat{K}} d_\pi \dim J(\pi) = \sum_{\pi \in \hat{K}} d_\pi \operatorname{mult}(\overline{\pi}, \rho) < \infty,$$

so there is a function $f \in L^2(K)$ with $\hat{f}(\pi) = P_\pi$ for all $\pi \in \hat{K}$, by Plancherel's Theorem. Moreover, $\langle f \rangle = V_J = \mathcal{H}_\rho$ by Theorem 4.2.7. Finally, Lemma 4.3.4 shows that

$$[f, f](\pi) = \hat{f}(\pi)^* \hat{f}(\pi) = P_\pi \qquad (\pi \in \hat{K}),$$

so $\{\rho(\xi) f\}_{\xi \in K}$ is a continuous Parseval frame for $\mathcal{H}_\rho$, by Theorem 4.4.3.

(ii) $\implies$ (i). This is trivial. $\square$

Two K-frames $\{\rho(\xi) f\}_{\xi \in K}$ and $\{\rho'(\xi) f'\}_{\xi \in K}$ are *unitarily equivalent* if there is a unitary $U \colon \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$ such that $U\rho(\xi) f = \rho'(\xi) f$ for all $\xi \in K$. Equivalently, U is a unitary equivalence of ρ and ρ' satisfying $Uf = f'$. We now classify K-frames up to unitary equivalence.

In the theorem below, we treat $L^2(K)$ as a Banach $*$-algebra under convolution. Thus, a *projection* in $L^2(K)$ is a function f with the property that $f = f * f = f^*$. Equivalently, it is a function f such that $\hat{f}(\pi)$ is an orthogonal projection for each

$\pi \in \hat{K}$. We also write

$$\mathcal{E}(K) = \{f \in L^2(K) : \hat{f}(\pi) = 0 \text{ for all but finitely many } \pi \in \hat{K}\}$$

for the space of trigonometric polynomials on K. Every projection in $L^2(K)$ belongs to $\mathcal{E}(K)$.

Theorem 4.5.3. *Up to unitary equivalence, K-frames are indexed by functions of positive type in $\mathcal{E}(K)$. If f is such a function, the associated frame is $\{L_\xi f\}_{\xi \in K}$. The same correspondence sets up a bijection between equivalence classes of Parseval K-frames and projections in $L^2(K)$.*

In the special case where K is finite, some aspects of this theorem appear implicitly in Vale and Waldron [59]. See also Han [34]. For Parseval K-frames, the fact that the generating function f is a projection implies that $V_f f = f * f^* = f$. By (4.31), the Gramian of the associated frame is the convolution operator $g \mapsto g * f$, which is orthogonal projection onto $\langle f \rangle$. In this sense, the theorem above may be compared with a result of Han and Larson [36, Corollary 2.7], which says that the correspondence between a frame and its Gramian induces a bijection between equivalence classes of Parseval frames indexed by a set I, and orthogonal projections on $\ell^2(I)$. For continuous frames, a similar result appears in [27, Proposition 2.1]. Lots of orthogonal projections on $L^2(K)$ correspond to continuous Parseval frames over K. The projections that correspond to K-frames are precisely those given by convolution.

Proof. We use the term *cyclic structure* for a pair (ρ, f) consisting of a cyclic representation ρ and a cyclic vector $f \in \mathcal{H}_\rho$. Call two cyclic structures (ρ, f) and (ρ', f') *equivalent* if there is a unitary equivalence between ρ and ρ' that maps f to

f'. This agrees with the notion of equivalence of K-frames. Given $f \in L^2(K)$, we will denote ρ_f for the subrepresentation of the regular representation on

$$\langle f \rangle = \{g \in L^2(K) : \operatorname{ran} \hat{g}(\pi) \subseteq \operatorname{ran} \hat{f}(\pi) \text{ for all } \pi \in \hat{K}\}.$$

Lemma 4.4.4 shows that

$$\{(\rho_f, f) : f \in L^2(K) \text{ is a function of positive type}\}$$

is a complete and irredundant set of cyclic structures, up to equivalence. For a fixed function $f \in L^2(K)$ of positive type, it only remains to show

$$\{L_\xi f\}_{\xi \in K} \text{ is a frame for } \langle f \rangle \iff \hat{f}(\pi) = 0 \text{ for all but finitely many } \pi \in \hat{K} \quad (4.35)$$

and

$$\{L_\xi f\}_{\xi \in K} \text{ is a Parseval frame for } \langle f \rangle \iff \quad (4.36)$$

$$\hat{f}(\pi) \text{ is an orthogonal projection for all } \pi \in \hat{K}.$$

The forward implication of (4.35) follows from Theorem 4.5.2, since

$$\operatorname{mult}(\overline{\pi}, \rho_f) = \operatorname{rank} \hat{f}(\pi) \qquad (\pi \in \hat{K}),$$

by (4.14). For the reverse implication, suppose that $\hat{f}(\pi) = 0$ for all but finitely many $\pi \in \hat{K}$. Then the operators $[f, f](\pi) = \hat{f}(\pi)^2$, $\pi \in \hat{K}$, have only finitely many nonzero eigenvalues between them, so $\{L_\xi f\}_{\xi \in K}$ is a continuous frame, by Theorem 4.4.3.

To prove (4.36), recall that $\hat{f}(\pi) \geq 0$ for all $\pi \in \hat{K}$, so the eigenvalues of $[f, f](\pi) = \hat{f}(\pi)^2$ are precisely the squares of the eigenvalues of $\hat{f}(\pi)$. By Theorem

4.4.3, $\{L_\xi f\}_{\xi\in K}$ is a continuous Parseval frame for $\langle f\rangle$ if and only if 0 and 1 are the only eigenvalues of $\hat{f}(\pi)$, $\pi\in\hat{K}$. Since the operators $\hat{f}(\pi)$ are self-adjoint, that happens if and only if each $\hat{f}(\pi)$ is an orthogonal projection. $\square$

Remark 4.5.4. A function $f\in L^2(K)$ is a projection if and only if $\hat{f}(\pi)$ is an orthogonal projection for each $\pi\in\hat{K}$. If we let $J(\pi)=\operatorname{ran}\hat{f}(\pi)\subseteq\mathcal{H}_\pi$, we see that Parseval K-frames can also be classified by range functions in $\{\mathcal{H}_\pi\}_{\pi\in\hat{K}}$ with the property that $J(\pi)=0$ for all but finitely many $\pi\in\hat{K}$.

Given a projection $f\in L^2(K)$, $\{L_\xi f\}_{\xi\in K}$ is a frame only for its closed linear span in $L^2(K)$, not necessarily for the whole space. This is troublesome in practice, where one usually wants coordinates for a frame in its "native domain". The corollary below gives such coordinates for every Parseval K-frame. When a matrix space $M_{m,n}(\mathbb{C})$ is treated as a Hilbert space below, its inner product is gotten from the natural identification with $\mathbb{C}^{mn}$.

Corollary 4.5.5. *For each $\pi\in\hat{K}$, choose an integer $r_\pi\in\{0,\dots,d_\pi\}$, in such a way that only finitely many $r_\pi\neq 0$. Choose an orthonormal basis for $\mathcal{H}_\pi$, and let $\pi_{i,j}\in C(K)$ be the corresponding matrix elements. Given $\xi\in K$, define $M_\xi(\pi)\in M_{r_\pi,d_\pi}(\mathbb{C})$ by*

$$M_\xi(\pi)=(\sqrt{d_\pi}\pi_{i,j}(\xi))_{1\le i\le r_\pi,1\le j\le d_\pi}.$$

Then $\{M_\xi\}_{\xi\in K}$ is a continuous Parseval frame for $\bigoplus_{\pi\in\hat{K}}M_{r_\pi,d_\pi}(\mathbb{C})$, and it is a K-frame when indexed $\{M_{\xi^{-1}}\}_{\xi\in K}$. Up to unitary equivalence, every Parseval K-frame is produced in this way.

Proof. First we will show that $\{M_{\xi^{-1}}\}_{\xi\in K}$ is a Parseval K-frame. For each $\pi\in\hat{K}$, let $e_1^\pi,\dots,e_{d_\pi}^\pi$ be the orthonormal basis for $\mathcal{H}_\pi$ used in the construction of $\{M_\xi\}_{\xi\in K}$. Let $P_\pi\in B(\mathcal{H}_\pi)$ be orthogonal projection onto $\operatorname{span}\{e_1^\pi,\dots,e_{r_\pi}^\pi\}$. By Plancherel's

Theorem, there is a projection $f \in L^2(K)$ with $\hat{f}(\pi) = P_\pi$ for each $\pi \in \hat{K}$. We are going to map

$$\langle f \rangle = \{g \in L^2(K) : \operatorname{ran} \hat{g}(\pi) \subseteq \operatorname{ran} P_\pi \text{ for each } \pi \in \hat{K}\}$$

unitarily onto $\bigoplus_{\pi \in \hat{K}} M_{r_\pi, d_\pi}(\mathbb{C})$ in a way that sends the Parseval K-frame $\{L_\xi f\}_{\xi \in K}$ to $\{M_{\xi^{-1}}\}_{\xi \in K}$.

For each $\pi \in \hat{K}$, assign $B(\mathcal{H}_\pi)$ the inner product $\langle A, B \rangle = d_\pi \langle A, B \rangle_{\mathcal{HS}}$, as in Plancherel's Theorem. There is a unitary $U_\pi \colon B(\mathcal{H}_\pi) \to M_{d_\pi}(\mathbb{C})$ that replaces each operator with $\sqrt{d_\pi}$ times its matrix over the chosen basis. Let

$$U \colon L^2(K) \to \bigoplus_{\pi \in \hat{K}} M_{d_\pi}(\mathbb{C})$$

be the unitary that follows the Fourier transform $\mathcal{F} \colon L^2(K) \to \bigoplus_{\pi \in \hat{K}} B(\mathcal{H}_\pi)$ by an application of U_π in every coordinate $\pi \in \hat{K}$. Given $\xi \in K$, the translation identity (4.1) shows that

$$(L_\xi f)\hat{}\,(\pi) = P_\pi \pi(\xi^{-1}) \qquad (\pi \in \hat{K}),$$

so the π-th coordinate of $U(L_\xi f)$ is the $d_\pi \times d_\pi$ matrix with $M_{\xi^{-1}}(\pi)$ in the top r_π rows and zeros in the bottom $d_\pi - r_\pi$ rows. Moreover,

$$U\langle f \rangle =$$

$$\{(A_\pi)_{\pi \in \hat{K}} \in \bigoplus_{\pi \in \hat{K}} M_{d_\pi}(\mathbb{C}) : \text{for each } \pi \in \hat{K},\ A_\pi \text{ has zeros in the bottom } d_\pi - r_\pi \text{ rows}\}.$$

Following U with the natural identification

$$U\langle f \rangle \cong \bigoplus_{\pi \in \hat{K}} M_{r_\pi, d_\pi}(\mathbb{C})$$

gives the desired unitary of $\langle f \rangle$ onto $\bigoplus_{\pi \in \hat{K}} M_{r_\pi, d_\pi}(\mathbb{C})$.

To see that *every* Parseval K-frame is produced in this way, reverse the procedure above for an arbitrary projection $f \in L^2(K)$. For each $\pi \in \hat{K}$, let $P_\pi = \hat{f}(\pi)$, let $r_\pi = \operatorname{rank} P_\pi$, and choose an orthonormal basis $e_1^\pi, \ldots, e_{d_\pi}^\pi$ for $\mathcal{H}_\pi$ in such a way that $\operatorname{ran} P_\pi = \operatorname{span}\{e_1^\pi, \ldots, e_{r_\pi}^\pi\}$. The Parsevel K-frame $\{M_{\xi^{-1}}\}_{\xi \in K}$ produced with these parameters is unitarily equivalent to $\{L_\xi f\}_{\xi \in K}$ through the isometries constructed above. $\square$

In the special case where K is finite and *abelian*, the frames described in Corollary 4.5.5 are precisely the "harmonic" frames made by deleting rows from a discrete Fourier transform (DFT) matrix. (See [58] for another proof that harmonic frames come from group actions.) While each finite abelian group can be used to make only finitely many Parseval frames in this way, a nonabelian group can make uncountably many inequivalent Parseval frames, since there are uncountably many projections in $L^2(K)$. (For finite groups, this was observed in [59].) Moreover, it is often possible to make *real* frames using nonabelian groups, as in the next example.

Example 4.5.6. Let $K = D_3$. Use notation as in Example 4.4.11. If we choose each r_π to be as large as possible in Corollary 4.5.5, we obtain the following tight frame:

$$\begin{pmatrix} (1) & (1) & (1) & (1) & (1) & (1) \\ \mathbf{(1)} & \mathbf{(1)} & \mathbf{(1)} & \mathbf{(-1)} & \mathbf{(-1)} & \mathbf{(-1)} \\ \begin{pmatrix} \boldsymbol{\sqrt{2}} & \mathbf{0} \\ 0 & \sqrt{2} \end{pmatrix} & \begin{pmatrix} \boldsymbol{\omega\sqrt{2}} & \mathbf{0} \\ 0 & \omega^2\sqrt{2} \end{pmatrix} & \begin{pmatrix} \boldsymbol{\omega^2\sqrt{2}} & \mathbf{0} \\ 0 & \omega\sqrt{2} \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \boldsymbol{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \boldsymbol{\omega\sqrt{2}} \\ \omega^2\sqrt{2} & 0 \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \boldsymbol{\omega^2\sqrt{2}} \\ \omega\sqrt{2} & 0 \end{pmatrix} \end{pmatrix}.$$

We can get another tight frame by deleting some of the rows:

$$\begin{pmatrix} \mathbf{(1)} & \mathbf{(1)} & \mathbf{(1)} & \mathbf{(-1)} & \mathbf{(-1)} & \mathbf{(-1)} \\ \begin{pmatrix} \boldsymbol{\sqrt{2}} & \mathbf{0} \end{pmatrix} & \begin{pmatrix} \boldsymbol{\omega\sqrt{2}} & \mathbf{0} \end{pmatrix} & \begin{pmatrix} \boldsymbol{\omega^2\sqrt{2}} & \mathbf{0} \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \boldsymbol{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \boldsymbol{\omega\sqrt{2}} \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \boldsymbol{\omega^2\sqrt{2}} \end{pmatrix} \end{pmatrix}.$$

This corresponds to choosing $r_1 = 0$ and $r_2 = r_3 = 1$. Collapsing the interior matrices gives a tight frame for $\mathbb{C}^3$:

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ \sqrt{2} & \omega\sqrt{2} & \omega^2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \omega\sqrt{2} & \omega^2\sqrt{2} \end{pmatrix}.$$

The frame bound is $\text{card}(D_3) = 6$; see Remark 4.3.2.

Representing the two-dimensional representation over a different basis gives a completely different frame. If we use

$$\pi_3(a) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad \text{and} \quad \pi_3(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and choose rows exactly as above, we obtain the tight frame

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ \sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -\sqrt{3/2} & \sqrt{3/2} & 0 & \sqrt{3/2} & -\sqrt{3/2} \end{pmatrix}.$$

This time we used real representations, so we got a tight frame for $\mathbb{R}^3$.

4.53. Disjointness properties

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces carrying frames $\Phi = \{f_i\}_{i\in I}$ and $\Psi = \{g_i\}_{i\in I}$, respectively. We say that Φ and Ψ are *disjoint* if $\{(f_i, g_i)\}_{i\in I}$ is a frame for $\mathcal{H} \oplus \mathcal{K}$. Disjoint frames were introduced independently by Balan [6] and by Han and Larson [36]. For a detailed study of disjoint *continuous* frames, see [27]. The corollary below says that K-frames from distinct isotypical components of ρ are always disjoint, and that every K-frame can be decomposed into disjoint frames in this way. This will be generalized for group frames with multiple generators in Corollary 4.6.10. Recall that $\mathcal{M}_\pi \subseteq \mathcal{H}_\rho$ denotes the isotypical component for $\pi \in \hat{K}$, and that $P_\pi \in B(\mathcal{H}_\rho)$ is orthogonal projection of $\mathcal{H}_\rho$ onto $\mathcal{M}_\pi$.

Corollary 4.5.7. *Fix a vector $f \in \mathcal{H}_\rho$ and constants A and B with $0 < A \leq B < \infty$. The following are equivalent.*

(i) $\{\rho(\xi)f\}_{\xi\in K}$ is a continuous frame for $\mathcal{H}_\rho$ with bounds A, B.

(ii) For each $\pi \in \hat{K}$, $\{\rho(\xi)P_\pi f\}_{\xi\in K}$ is a continuous frame for $\mathcal{M}_\pi$ with bounds A, B.

Proof. For each $\pi \in \hat{K}$ and each $g \in \mathcal{H}_\rho$, Proposition 4.4.8 shows that

$$[P_\pi f, P_\pi g](\sigma) = [P_\pi f, g](\sigma) = \begin{cases} [f,g](\overline{\pi}), & \text{if } \sigma = \overline{\pi} \\ 0, & \text{if } \sigma \neq \overline{\pi}. \end{cases} \tag{4.37}$$

Taking $g = f$ above, we see that f satisfies condition (ii) of Theorem 4.4.3 if and only if each $P_\pi f$ does the same. It remains to show that $\langle f \rangle = \mathcal{H}_\rho$ if and only if $\langle P_\pi f \rangle = \mathcal{M}_\pi$ for each $\pi \in \hat{K}$.

If $\langle f \rangle \neq \mathcal{H}_\rho$, then we can find a nonzero vector $g \in \mathcal{H}_\rho$ with $[f,g] = 0$, by Proposition 4.4.5(vi). Find $\pi \in \hat{K}$ for which $P_\pi g \neq 0$. Then (4.37) shows that $[P_\pi f, P_\pi g] = 0$, so that $P_\pi g \perp \langle P_\pi f \rangle$. Thus, $\langle P_\pi f \rangle \neq \mathcal{M}_\pi$.

Conversely, if there is some $\pi \in \hat{K}$ for which $\langle P_\pi f \rangle \neq \mathcal{M}_\pi$, then there is a nonzero vector $g \in \mathcal{M}_\pi$ with $0 = [P_\pi f, g] = [f, P_\pi g] = [f,g]$. Hence, $g \perp \langle f \rangle$, and $\langle f \rangle \neq \mathcal{H}_\rho$. $\square$

Recall that ρ is *multiplicity free* when all of its isotypical components are irreducible. Equivalently, this means that $\mathrm{mult}(\pi, \rho) \in \{0, 1\}$ for all $\pi \in \hat{K}$. Corollary 4.5.7 leads to an extension of Example 4.4.10 for multiplicity free representations.

Corollary 4.5.8. *Suppose ρ is multiplicity free. Let $E = \{\pi \in \hat{K} : \mathrm{mult}(\pi, \rho) \neq 0\}$. For a nonzero vector $f \in \mathcal{H}_\rho$, the following are equivalent.*

(i) $\{\rho(\xi) f\}_{\xi \in K}$ *is a tight frame for* $\mathcal{H}_\rho$.

(ii) *For any* $\pi, \sigma \in E$, $\|P_\pi f\|^2 / d_\pi = \|P_\sigma f\|^2 / d_\sigma$.

When this happens, the optimal frame bound is the common value of $\|P_\pi f\|^2 / d_\pi$ *for* $\pi \in E$.

In the special case where K is *finite*, the equivalence of (i) and (ii) above can be deduced from [58, Theorem 6.18].

We mention just one of a myriad applications for Corollary 4.5.7. An action of a group G on a set X is called *2-transitive* when the following holds: for every two pairs $(x, y), (w, z) \in X \times X$ with $x \neq y$ and $w \neq z$, there is a single group element $g \in G$ with $g \cdot x = w$ and $g \cdot y = z$.

Corollary 4.5.9. *Let G be a finite group acting on a finite set X with an action that is 2-transitive. Fix a nonzero vector $f = (f_x)_{x \in X} \in \ell^2(X)$. Then $\{(f_{g \cdot x})_{x \in X} : g \in G\}$ is a tight frame for $\ell^2(X)$ if and only if*

$$\left| \sum_{x \in X} f_x \right|^2 = \sum_{x \in X} |f_x|^2. \tag{4.38}$$

Proof. The statement is trivial when X is a singleton, so we may assume that X has more than one point. Let ρ be the unitary representation of G on $\ell^2(X)$ associated with the action of G. Namely, for $g \in G$ and $\psi = (\psi_x)_{x \in X} \in \ell^2(X)$, we define $\rho(g)\psi = (\psi_{g^{-1} \cdot x})_{x \in X}$. By [45, Corollary 29.10], ρ is multiplicity free with two isotypical components,

$$\mathcal{M}_1 = \big\{ (\psi_x)_{x \in X} \in \ell^2(X) : \psi_x = \psi_y \text{ for all } x, y \in X \big\}$$

and

$$\mathcal{M}_2 = \left\{ (\psi_x)_{x \in X} \in \ell^2(X) : \sum_{x \in X} \psi_x = 0 \right\}.$$

Let P_j be orthogonal projection of $\ell^2(X)$ onto $\mathcal{M}_j$, for $j = 1, 2$. If we denote

$$\overline{f} = \frac{1}{|X|} \sum_{x \in X} f_x,$$

then $P_1 f = (\overline{f})_{x\in X}$, and $P_2 f = (f_x - \overline{f})_{x\in X}$. In particular,

$$\|P_1 f\|^2 = \frac{1}{|X|}\left|\sum_{x\in X} f_x\right|^2.$$

By Corollary 4.5.8, the orbit of f under ρ is a tight frame for $\ell^2(X)$ if and only if $\|P_1 f\|^2 = \|P_2 f\|^2/(|X|-1)$, if and only if $|X|\cdot\|P_1 f\|^2 = \|P_2 f\|^2 + \|P_1 f\|^2 = \|f\|^2$, if and only if

$$\left|\sum_{x\in X} f_x\right|^2 = \sum_{x\in X}|f_x|^2. \qquad \square$$

The proof indicates a simple and universal method for constructing the generating vector f. Let $\varphi \in \ell^2(X)$ be the all-ones vector. Fix any nonzero vector $\psi \in \ell^2(X)$ with $\sum_{x\in X}\psi_x = 0$, and scale it so that $\|\psi\|^2 = |X|^2 - |X|$. Then $f = \varphi + \psi$ generates a tight frame for $\ell^2(X)$, by Corollary 4.5.8. Up to scaling, every vector satisfying (4.38) is produced in this way.

Example 4.5.10. The action of the symmetric group S_n on the set with n elements is 2-transitive. Thus, Corollary 4.5.9 and the comment above explain how to make a unit norm tight frame of $n!$ vectors in $\mathbb{C}^n$ just by permuting the entries of a single vector.

4.6. Group frames with multiple generators

The last two sections focused on frames generated by a single vector $f \in \mathcal{H}_\rho$. We now consider frames with *multiple* generators. For a countable family $\mathscr{A} = \{f_j\}_{j\in I} \subseteq \mathcal{H}_\rho$, this means that we will determine precise (and simple) conditions under which the orbit $\{\rho(\xi)f_j\}_{j\in I,\xi\in K}$ forms a continuous frame for $\mathcal{H}_\rho$. In the course of doing so, we will classify the invariant subspaces of $\mathcal{H}_\rho$ in terms of range functions.

Despite significant interest in the problem, very little has been done in the area of group frames with multiple generators. The most fruitful area has been frames generated by translations, mostly with abelian groups [10, 11, 13, 14, 42, 49] but in at least one case with nonabelian [19]. In the setting of discrete nonabelian groups, Hernández and his collaborators [9] have recently developed an abstract machinery to handle frames with multiple generators for a special class of unitary representations. For finite groups and *tight* frames, Vale and Waldron [60] recently broke through the single generator barrier, with a neat condition in terms of norms and orthogonality of the generating vectors. These few papers provide the state of the art.

Our main result is a duality theorem unifying the work of Vale and Waldron with classical duality of frames and Riesz sequences, simultaneously extending their results to non-tight frames and actions by compact groups. Here we pull ahead of the abelian setting. As far as the author knows, there is nothing of this kind in the literature for LCA groups. Once again, we hope that by illuminating the situation for nonabelian compact groups, we can set a path for further research on representations of general locally compact groups.

Our notation and assumptions are as follows. Let K and ρ be as in the previous sections. Since K is compact, it is always possible to decompose $\mathcal{H}_\rho$ as a direct sum of irreducible invariant subspaces. Our main assumption is that *this has already been done*. For $\pi \in \hat{K}$, we let $m_\pi = \text{mult}(\pi, \rho)$. We write $\pi^{\oplus m_\pi}$ for the direct sum of m_π copies of π, which acts on $\mathcal{H}_\pi^{\oplus m_\pi}$. Without loss of generalty, we may assume that

$$\rho = \bigoplus_{\pi \in \hat{K}} \pi^{\oplus m_\pi},$$

and that

$$\mathcal{H}_\rho = \bigoplus_{\pi \in \hat{K}} \mathcal{H}_\pi^{\oplus m_\pi}.$$

We warn that some of the multiplicities m_π may be infinite, but since $\mathcal{H}_\rho$ is separable, they must all be countable.

Fix the following notation. Let $\mathscr{A} = \{f_j\}_{j \in I} \subseteq \mathcal{H}_\rho$ be a countable family of vectors. We write $f_j = (f_j^\pi)_{\pi \in \hat{K}} \in \mathcal{H}_\rho$, with $f_j^\pi = (f_{i,j}^\pi)_{i=1}^{m_\pi} \in \mathcal{H}_\pi^{\oplus m_\pi}$. We also denote

$$E(\mathscr{A}) = \{\rho(\xi) f_j\}_{j \in I, \xi \in K}$$

for the orbit of $\mathscr{A}$ under ρ. Formally, $E(\mathscr{A})$ should be interpreted as a set with multiplicities, or more accurately, as a mapping $I \times K \to \mathcal{H}_\rho$. Finally, we let

$$S(\mathscr{A}) = \overline{\operatorname{span}}\{\rho(\xi) f_j : j \in I, \xi \in K\}$$

be the invariant subspace generated by $\mathscr{A}$.

Our notation is meant to suggest that $\mathscr{A}$ is a kind of matrix. For each $\pi \in \hat{K}$, we define

$$\mathscr{A}(\pi) = (f_{i,j}^\pi)_{1 \leq i \leq m_\pi, j \in I},$$

which is a (possibly infinite) matrix with entries in $\mathcal{H}_\pi$. The number of rows equals m_π, and the number of columns equals $\operatorname{card}(\mathscr{A})$. For instance, if $\mathscr{A}$ were finite with $I = \{1, \ldots, N\}$, we would have

$$\mathscr{A}(\pi) = \begin{pmatrix} | & | & \cdots & | \\ f_1^\pi & f_2^\pi & \cdots & f_N^\pi \\ | & | & \cdots & | \end{pmatrix}.$$

If we now imagine the matrices $\mathscr{A}(\pi)$ stacked vertically, then the j-th column of the resulting "matrix" precisely describes the direct sum decomposition of $f_j \in \mathscr{A}$.

We remind the reader that a *Riesz sequence* in a Hilbert space $\mathcal{H}$ is a sequence of vectors $\{f_i\}_{i\in J} \subseteq \mathcal{H}$ for which there are constants $0 < A \leq B < \infty$ such that, whenever $(c_i) \in \ell^2(J)$ has finite support,

$$A \sum_{i\in J} |c_i|^2 \leq \left\| \sum_{i\in J} c_i f_i \right\|^2 \leq B \sum_{i\in J} |c_i|^2.$$

Once this inequality holds for those $(c_i) \in \ell^2(J)$ with finite support, it automatically holds for arbitrary $(c_i) \in \ell^2(J)$. Our main result, below, says that the frame properties of the orbit of the "columns" of $\mathscr{A}$ can be read from the Riesz properties of the *rows*.

Theorem 4.6.1. *The following are equivalent for constants A and B with $0 < A \leq B < \infty$.*

(i) The orbit $E(\mathscr{A}) = \{\rho(\xi)f_j\}_{j\in I,\xi\in K}$ is a continuous frame for $\mathcal{H}_\rho$ with bounds A, B.

(ii) For every $\pi \in \hat{K}$, the rows of $\mathscr{A}(\pi)$ belong to $\mathcal{H}_\pi^{\oplus I}$, where they form a Riesz sequence with bounds $d_\pi A, d_\pi B$.

This will actually be a corollary of a more general theorem. Theorem 4.6.6 (infra) gives conditions for $E(\mathscr{A})$ to form a continuous frame for a general invariant subspace of $\mathcal{H}_\rho$.

Example 4.6.2. Here are four special cases of Theorem 4.6.1.

(1) When K is the trivial group and ρ is the trivial action of K on $\mathbb{C}$, we recover the usual duality theorem for frames and Riesz sequences, which says that the columns of a matrix $M \in M_{m,n}(\mathbb{C})$ form a frame for $\mathbb{C}^m$ if and only if the rows of M form a

Riesz sequence in $\mathbb{C}^n$. Moreover, the bounds of the frame and the Riesz sequence are the same.

(2) When $\mathscr{A}$ has a single vector f and ρ is irreducible, there is only one matrix $\mathscr{A}(\pi)$ to consider, namely $\mathscr{A}(\rho) = (f)$. Obviously its rows form a Riesz sequence with upper and lower bounds both equal to $\|f\|^2$, so the orbit $\{\rho(\xi)f\}_{\xi\in K}$ is a tight frame for $\mathcal{H}_\rho$ with bound $\|f\|^2 / (\dim \mathcal{H}_\rho)$. This is the conclusion of Example 4.4.10.

(3) More generally, when ρ is multiplicity free, we can easily recover Corollary 4.5.8.

(4) Taking $A = B$ in Theorem 4.6.1, we see that $E(\mathscr{A})$ is a tight frame for $\mathcal{H}_\rho$ with bound A if and only if the rows of each matrix $\mathscr{A}(\pi)$ form an orthogonal sequence of vectors in $\mathcal{H}_\pi^{\oplus I}$, with each vector's norm equal to $\sqrt{d_\pi A}$. That is,

$$\sum_{j\in I} \langle f^\pi_{i_1,j}, f^\pi_{i_2,j} \rangle = \delta_{i_1,i_2} \cdot d_\pi A.$$

In the case where K and $\mathscr{A}$ are both *finite*, this is a result of Vale and Waldron [60, Theorem 2.8].

Remark 4.6.3. Neither the group K nor the representation ρ play a prominent role in condition (ii) of Theorem 4.6.1, except to provide conditions on the direct sum decomposition $\mathcal{H}_\rho = \bigoplus_{\pi\in\hat{K}} \mathcal{H}_\pi^{\oplus m_\pi}$. Suppose, then, that G is *another* compact group acting on $\mathcal{H}_\rho$ with a representation η that admits the same decomposition of $\mathcal{H}_\rho$ as a direct sum of irreducible invariant subspaces. Then the orbit of $\mathscr{A}$ under the action of ρ is a frame for $\mathcal{H}_\rho$ if and only if the orbit under the action of η is, too. Moreover, the frame bounds are the same in both cases.

While this may seem surprising at first, it is really an extension of a well-known phenomenon. After all, any nonzero vector $f \in \mathcal{H}_\rho$ generates a tight frame when ρ

acts irreducibly, and this mild condition ($f \neq 0$) has nothing to do with K or the particular irreducible representation ρ. As we have seen, this is a special case of Theorem 4.6.1.

4.61. Classification of invariant subspaces

From a technical perspective, we can always find an encompassing group $G \supseteq K$ for which $\mathcal{H}_\rho$ embeds into $L^2(G)$ as a K-invariant subspace, with ρ turning into left translation. (See Theorem 4.4.6.) In this sense, Theorem 4.3.3 on frames generated by translations already gives a complete characterization of group frames with multiple generators. In practice, however, it may be tedious to unravel this characterization through the embedding $\mathcal{H}_\rho \to L^2(G)$. Instead of following that route, we will now try to recreate the program of Sections 4.1–4.3 from scratch. Namely, we will give a range function characterization of the invariant subspaces of $\mathcal{H}_\rho$, and then we will use that characterization to deduce Theorem 4.6.1.

To begin our program, we need a substitute for the Zak transform. Fix $\pi \in \hat{K}$, and associate each sequence $\Phi = (\phi_i)_{i=1}^{m_\pi} \in \mathcal{H}_\pi^{\oplus m_\pi}$ with its analysis operator $T_\pi \Phi \colon \mathcal{H}_\pi \to \ell^2_{m_\pi}$, which is given by

$$[T_\pi \Phi](\psi) = (\langle \psi, \phi_i \rangle)_{i=1}^{m_\pi} \qquad (\psi \in \mathcal{H}_\pi).$$

Then $T_\pi \colon \mathcal{H}_\pi^{\oplus m_\pi} \to \mathcal{HS}(\mathcal{H}_\pi, \ell^2_{m_\pi})$ is a *conjugate*-linear unitary. To see this, consider the composition of isomorphisms

$$\mathcal{H}_\pi^{\oplus m_\pi} \cong \mathcal{H}_\pi \otimes \ell^2_{m_\pi} \cong \mathcal{HS}(\mathcal{H}_\pi, \ell^2_{m_\pi}),$$

the last of which is conjugate linear (see [26, Section 7.3]). Letting π run through $\hat{K}$, we obtain a conjugate-linear unitary

$$T\colon \mathcal{H}_\rho \to \bigoplus_{\pi\in\hat{K}} \mathcal{HS}(\mathcal{H}_\pi, \ell^2_{m_\pi})$$

given by

$$T(g_\pi)_{\pi\in\hat{K}} = (T_\pi g_\pi)_{\pi\in\hat{K}} \qquad ((g_\pi)_{\pi\in\hat{K}} \in \bigoplus_{\pi\in\hat{K}} \mathcal{H}_\pi^{\oplus m_\pi} = \mathcal{H}_\rho).$$

If we write $g_\pi = (g_i^\pi)_{i=1}^{m_\pi} \in \mathcal{H}_\pi^{\oplus m_\pi}$, then the simple formula $\langle \phi, \pi(\xi) g_i^\pi\rangle = \langle \pi(\xi^{-1})\phi, g_i^\pi\rangle$ gives the key identity

$$(T\rho(\xi)g)(\pi) = (Tg)(\pi)\cdot\pi(\xi^{-1}) \qquad (g\in\mathcal{H}_\rho,\ \xi\in K,\ \pi\in\hat{K}). \tag{4.39}$$

This will serve as our substitute for the Zak transform's translation property (4.8).

A careful reading of Section 4.2 shows that we used only two properties of the Zak transform: the translation property (4.8), and the fact that Z is unitary. In the current setting, we can therefore leverage the intertwining property (4.39) to classify invariant subspaces of $\mathcal{H}_\rho$ in terms of range functions. Let J be a range function in $\{\ell^2_{m_\pi}\}_{\pi\in\hat{K}}$, and let

$$V_J = \{(g_\pi)_{\pi\in\hat{K}} \in \mathcal{H}_\rho : \text{for each } \pi\in\hat{K},\ \operatorname{ran} T_\pi g_\pi \subseteq J(\pi)\}.$$

Equivalently,

$$TV_J = \bigoplus_{\pi\in\hat{K}} \mathcal{HS}(\mathcal{H}_\pi, J(\pi)).$$

By (4.39), V_J is an invariant subspace of $\mathcal{H}_\rho$. In fact, a trivial modification of the proof of Theorem 4.2.2 shows that every invariant subspace of $\mathcal{H}_\rho$ takes this form. Explicitly, we have the following.

Theorem 4.6.4. *The mapping $J \mapsto V_J$ is a bijection between range functions in $\{\ell^2_{m_\pi}\}_{\pi \in \hat{K}}$ and invariant subspaces of $\mathcal{H}_\rho$.*

In further analogy with the range function analysis of Section 4.2, it is easy to see that the correspondence $J \mapsto V_J$ preserves direct sum decompositions. This leads to the following analogue of Theorem 4.2.9.

Theorem 4.6.5. *Let J be a range function in $\{\ell^2_{m_\pi}\}_{\pi \in \hat{K}}$. Choose an orthonormal basis $\{e_i^\pi\}_{i \in I_\pi}$ for each $J(\pi)$, $\pi \in \hat{K}$. For each $\pi \in \hat{K}$ and $i \in I_\pi$, let $V_{\pi,i}$ be the space of $(g_\sigma)_{\sigma \in \hat{K}} \in \mathcal{H}_\rho$ such that $g_\sigma = 0$ for $\sigma \neq \pi$, and such that $g_\pi = (g_j^\pi)_{j=1}^{m_\pi}$ satisfies $(\langle \phi, g_j^\pi \rangle)_{j=1}^{m_\pi} = c_\phi e_i^\pi$ for every $\phi \in \mathcal{H}_\pi$, where c_ϕ is a scalar. Then $V_{\pi,i}$ is an irreducible invariant subspace of $\mathcal{H}_\rho$, and*

$$V_J = \bigoplus_{\pi \in \hat{K}} \bigoplus_{i \in I_\pi} V_{\pi,i}.$$

Moreover, every decomposition of V_J as a direct sum of irreducible subspaces occurs in this way.

When J is the range function with $J(\pi) = \ell^2_{m_\pi}$ for every $\pi \in \hat{K}$, the theorem above describes every possible decomposition of $\mathcal{H}_\rho$ as a direct sum of irreducibles. Remember that our operating assumption is that we can find *one* such decomposition. Thus, knowing one decomposition is enough to describe them all (and very simply, at that).

4.62. Duality for frames with multiple generators

Now we can prove our main theorem on group frames with multiple generators. Remember our interpretation of $\mathscr{A}$ as a kind of matrix, with the vectors $f_j \in \mathscr{A}$ appearing as the "columns". It turns out that the frame properties of the orbit of the "columns" of $\mathscr{A}$ can be read from a Riesz-like property on the *rows*.

Theorem 4.6.6. *Let J be a range function in $\{\ell^2_{m_\pi}\}_{\pi\in\hat{K}}$, and assume that $\mathscr{A} \subseteq V_J$. For constants A and B with $0 < A \leq B < \infty$, the following are equivalent.*

(i) $E(\mathscr{A})$ is a continuous frame for V_J with bounds A, B. That is,

$$A\,\|g\|^2 \leq \sum_{j\in I} \int_K |\langle g, \rho(\xi) f_j\rangle|^2 \, d\xi \leq B\,\|g\|^2 \qquad (g \in V_J).$$

(ii) For every $\pi \in \hat{K}$ and every sequence $(c_i)_{i=1}^{m_\pi} \in J(\pi) \subseteq \ell^2_{m_\pi}$,

$$d_\pi A \sum_{i=1}^{m_\pi} |c_i|^2 \leq \sum_{j\in I} \left\| \sum_{i=1}^{m_\pi} c_i f^\pi_{i,j} \right\|^2 \leq d_\pi B \sum_{i=1}^{m_\pi} |c_i|^2.$$

Proof. Fix $g, h \in \mathcal{H}_\rho$. We will denote $g = (g_\pi)_{\pi\in\hat{K}}$, with $g_\pi \in \mathcal{H}_\pi^{\oplus m_\pi}$, and $g_\pi = (g_i^\pi)_{i=1}^{m_\pi}$, with $g_i^\pi \in \mathcal{H}_\pi$. We use a similar notation for h. For each $\pi \in \hat{K}$, fix an orthonormal basis $e_1^\pi, \ldots, e_{d_\pi}^\pi$ for $\mathcal{H}_\pi$, and let $\pi_{i,j} \in C(K)$ be the corresponding matrix elements. We are going to decompose $V_h g \in L^2(K)$ in the orthonormal basis $\{\sqrt{d_\pi}\pi_{i,j} : \pi \in \hat{K}, 1 \leq i, j \leq d_\pi\}$.

For any $\xi \in K$, we can use (4.39) and the fact that T is a conjugate-linear unitary to write

$$\langle g, \rho(\xi) h\rangle = \langle T\rho(\xi)h, Tg\rangle = \sum_{\pi\in\hat{K}} \langle (T_\pi h_\pi)\pi(\xi^{-1}), T_\pi g_\pi\rangle_{\mathcal{HS}}$$

$$= \sum_{\pi\in\hat{K}} \sum_{k=1}^{d_\pi} \langle (T_\pi h_\pi)\pi(\xi^{-1})e_k^\pi, (T_\pi g_\pi)e_k^\pi \rangle = \sum_{\pi\in\hat{K}} \sum_{k=1}^{d_\pi} \sum_{i=1}^{m_\pi} \langle \pi(\xi^{-1})e_k^\pi, h_i^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle$$

$$= \sum_{\pi\in\hat{K}} \sum_{k=1}^{d_\pi} \sum_{i=1}^{m_\pi} \sum_{l=1}^{d_\pi} \langle e_l^\pi, h_i^\pi \rangle \langle \pi(\xi^{-1})e_k^\pi, e_l^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle$$

$$= \sum_{\pi\in\hat{K}} \sum_{k=1}^{d_\pi} \sum_{i=1}^{m_\pi} \sum_{l=1}^{d_\pi} \langle e_l^\pi, h_i^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle \overline{\pi_{k,l}(\xi)}.$$

By using the inequalities $|\langle e_l^\pi, h_i^\pi \rangle|^2 \le \|h_i^\pi\|^2$, $|\langle g_i^\pi, e_k^\pi \rangle|^2 \le \|g_i^\pi\|^2$, and $|\pi_{k,l}(\xi)| \le 1$, one can easily show that

$$\sum_{i=1}^{m_\pi} |\langle e_l^\pi, h_i^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle \overline{\pi_{k,l}(\xi)}| \le \|h_\pi\| \, \|g_\pi\| < \infty \qquad (\pi \in \hat{K};\ k, l = 1, \ldots, d_\pi).$$

Thus, we can reorder the sum above to write

$$\langle g, \rho(\xi)h \rangle = \sum_{\pi\in\hat{K}} \sum_{k,l=1}^{d_\pi} \left(\frac{1}{\sqrt{d_\pi}} \sum_{i=1}^{m_\pi} \langle e_l^\pi, h_i^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle \right) \sqrt{d_\pi}\, \overline{\pi}_{k,l}(\xi).$$

We want to apply the Peter-Weyl Theorem to conclude that

$$\int_K |\langle g, \rho(\xi)h \rangle|^2 d\xi = \sum_{\pi\in\hat{K}} \frac{1}{d_\pi} \sum_{k,l=1}^{d_\pi} \left| \sum_{i=1}^{m_\pi} \langle e_l^\pi, h_i^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle \right|^2. \tag{4.40}$$

To justify (4.40), it suffices to prove the sum on the right is finite. To see this is the case, first observe that for $\pi \in \hat{K}$,

$$\sum_{i=1}^{m_\pi} \langle e_l^\pi, h_i^\pi \rangle \langle g_i^\pi, e_k^\pi \rangle = \langle (T_\pi h_\pi)e_l^\pi, (T_\pi g_\pi)e_k^\pi \rangle \qquad (k, l = 1, \ldots, d_\pi).$$

Denoting $\|\cdot\|_{\mathrm{op}}$ for the operator norm, we have

$$\frac{1}{d_\pi}\sum_{k,l=1}^{d_\pi}|\langle (T_\pi h_\pi)e_l^\pi,(T_\pi g_\pi)e_k^\pi\rangle|^2=\frac{1}{d_\pi}\sum_{l=1}^{d_\pi}\|(T_\pi g_\pi)^*(T_\pi h_\pi)e_l^\pi\|^2\le\|(T_\pi g_\pi)^*(T_\pi h_\pi)\|_{\mathrm{op}}^2$$

$$\le\|T_\pi g_\pi\|_{\mathrm{op}}^2\,\|T_\pi h_\pi\|_{\mathrm{op}}^2\le\|T_\pi g_\pi\|_{\mathcal{HS}}^2\,\|T_\pi h_\pi\|_{\mathcal{HS}}^2.$$

Since $\|g\|^2=\sum_{\pi\in\hat{K}}\|T_\pi g_\pi\|_{\mathcal{HS}}^2$, there is some $M>0$ such that $\|T_\pi g_\pi\|_{\mathcal{HS}}^2\le M$ for all $\pi\in\hat{K}$. Hence,

$$\sum_{\pi\in\hat{K}}\frac{1}{d_\pi}\sum_{k,l=1}^{d_\pi}\left|\sum_{i=1}^{m_\pi}\langle e_l^\pi,h_i^\pi\rangle\langle g_i^\pi,e_k^\pi\rangle\right|^2\le\sum_{\pi\in\hat{K}}\|T_\pi g_\pi\|_{\mathcal{HS}}^2\,\|T_\pi h_\pi\|_{\mathcal{HS}}^2\le M\,\|h\|^2<\infty.$$

This proves (4.40).

We continue by refining the expression on the right side of (4.40) even further. For $\pi\in\hat{K}$ and $k\in\{1,\dots,d_\pi\}$, we claim that

$$\sum_{l=1}^{d_\pi}\left|\sum_{i=1}^{m_\pi}\langle e_l^\pi,h_i^\pi\rangle\langle g_i^\pi,e_k^\pi\rangle\right|^2=\left\|\sum_{i=1}^{m_\pi}\langle e_k^\pi,g_i^\pi\rangle h_i^\pi\right\|^2. \tag{4.41}$$

Indeed, we can write

$$\sum_{l=1}^{d_\pi}\left|\sum_{i=1}^{m_\pi}\langle e_l^\pi,h_i^\pi\rangle\langle g_i^\pi,e_k^\pi\rangle\right|^2=\sum_{l=1}^{d_\pi}\sum_{i=1}^{m_\pi}\sum_{j=1}^{m_\pi}\langle e_l^\pi,h_i^\pi\rangle\langle g_i^\pi,e_k^\pi\rangle\langle h_j^\pi,e_l^\pi\rangle\langle e_k^\pi,g_j^\pi\rangle$$

$$=\sum_{i=1}^{m_\pi}\sum_{j=1}^{m_\pi}\langle g_i^\pi,e_k^\pi\rangle\langle e_k^\pi,g_j^\pi\rangle\sum_{l=1}^{d_\pi}\langle e_l^\pi,h_i^\pi\rangle\langle h_j^\pi,e_l^\pi\rangle=\sum_{i=1}^{m_\pi}\sum_{j=1}^{m_\pi}\langle g_i^\pi,e_k^\pi\rangle\langle e_k^\pi,g_j^\pi\rangle\langle h_j^\pi,h_i^\pi\rangle$$

$$=\sum_{i=1}^{m_\pi}\sum_{j=1}^{m_\pi}\langle\langle e_k^\pi,g_j^\pi\rangle h_j^\pi,\langle e_k^\pi,g_i^\pi\rangle h_i^\pi\rangle.$$

Since $|\langle e_k^\pi, g_i^\pi\rangle|^2 \leq \|g_i^\pi\|^2$, one can show that $\sum_{i=1}^{m_\pi} \|\langle e_k^\pi, g_i^\pi\rangle h_i^\pi\| \leq \|g_\pi\| \, \|h_\pi\| < \infty$. Hence the sum $\sum_{i=1}^{m_\pi} \langle e_k^\pi, g_i^\pi\rangle h_i^\pi$ converges in $\mathcal{H}_\pi$. That means we can move the sums inside the inner product above. This gives (4.41).

Combining (4.40) with (4.41), and letting h run through $\mathscr{A}$, we obtain the critical identity

$$\sum_{j\in I} \int_K |\langle g, \rho(\xi) f_j\rangle|^2 d\xi = \sum_{\pi\in\hat{K}} \sum_{k=1}^{d_\pi} \frac{1}{d_\pi} \sum_{j\in I} \left\| \sum_{i=1}^{m_\pi} \langle e_k^\pi, g_i^\pi\rangle f_{i,j}^\pi \right\|^2 \qquad (g \in \mathcal{H}_\rho). \tag{4.42}$$

Meanwhile,

$$\|g\|^2 = \sum_{\pi\in\hat{K}} \sum_{k=1}^{d_\pi} \sum_{i=1}^{m_\pi} |\langle e_k^\pi, g_i^\pi\rangle|^2 \qquad (g \in \mathcal{H}_\rho). \tag{4.43}$$

The rest of the proof comes easily. If $g \in V_J$, then $(\langle e_k^\pi, g_i^\pi\rangle)_{i=1}^{m_\pi} \in \operatorname{ran} T_\pi g_\pi \subseteq J(\pi)$ for every $\pi \in \hat{K}$ and every $k \in \{1, \ldots, d_\pi\}$. Thus, (ii) implies (i).

Now assume (i) holds. Fix $\pi \in \hat{K}$, and let $(c_i)_{i=1}^{m_\pi} \in J(\pi)$ be arbitrary. Define $g \in \mathcal{H}_\rho$ by

$$g_i^\sigma = \begin{cases} \overline{c_i} e_1^\pi, & \text{if } \sigma = \pi \\ 0, & \text{if } \sigma \neq \pi \end{cases} \qquad (\sigma \in \hat{K},\ 1 \leq i \leq m_\sigma).$$

Then

$$\|g\|^2 = \sum_{i=1}^{m_\pi} |c_i|^2,$$

while (4.42) gives

$$\sum_{j\in I} \int_K |\langle g, \rho(\xi) f_j\rangle|^2 d\xi = \frac{1}{d_\pi} \sum_{j\in I} \left\| \sum_{i=1}^{m_\pi} c_i f_{i,j}^\pi \right\|^2.$$

Since $\operatorname{ran}(Tg)(\sigma) \subseteq J(\sigma)$ for each $\sigma \in \hat{K}$, (i) applies to tell us that

$$A \sum_{i=1}^{m_\pi} |c_i|^2 \leq \frac{1}{d_\pi} \sum_{j \in I} \left\| \sum_{i=1}^{m_\pi} c_i f_{i,j}^\pi \right\|^2 \leq B \sum_{i=1}^{m_\pi} |c_i|^2.$$

This is (ii). $\square$

Corollary 4.6.7. *If $E(\mathscr{A})$ is a continuous frame for $S(\mathscr{A})$, then every row of $\mathscr{A}(\pi)$ belongs to $\mathcal{H}_\pi^{\oplus I}$, for every $\pi \in \hat{K}$.*

Proof. Fix $\pi \in \hat{K}$. Let $i_0 \in \{1, \ldots, m_\pi\}$ when $m_\pi < \infty$ and $i_0 \in \mathbb{N}$ when $m_\pi = \infty$. Denote $\delta_{i_0} \in \ell^2_{m_\pi}$ for the vector with a 1 in the i_0-th coordinate and 0 in all others, and let J be the range function given by

$$J(\sigma) = \begin{cases} \operatorname{span}\{\delta_{i_0}\}, & \text{if } \sigma = \pi \\ \{0\}, & \text{if } \sigma \neq \pi. \end{cases}$$

Then V_J is the i_0-th summand of $\mathcal{H}_\pi^{\oplus m_\pi} \subseteq \mathcal{H}_\rho$. Let $P \colon S(\mathscr{A}) \to V_J$ be the restriction to $S(\mathscr{A})$ of the orthogonal projection $\mathcal{H}_\rho \to V_J$. Since V_J is an invariant subspace of $\mathcal{H}_\rho$, P commutes with $\rho(\xi)$ for every $\xi \in K$, and the range of P is an invariant subspace of V_J.

Since V_J is irreducible, one of two things must happen: either the range of P is zero, or it is all of V_J. In the former case, we have $f_{i_0,j}^\pi = 0$ for all $j \in I$, so that the i_0-th row of $\mathscr{A}(\pi)$ equals $0 \in \mathcal{H}_\pi^{\oplus I}$. In the latter case, $\{P\rho(\xi) f_j\}_{j \in I, \xi \in K} = \{\rho(\xi) P f_j\}_{j \in I, \xi \in K}$ is a continuous frame for V_J. Say the upper bound is $B > 0$. Applying Theorem 4.6.6 with δ_{i_0} in place of $(c_i)_{i=1}^{m_\pi}$, we find that

$$\sum_{j \in I} \left\| f_{i_0,j}^\pi \right\|^2 \leq B d_\pi < \infty.$$

Thus, $(f_{i_0,j})_{j\in I} \in \mathcal{H}_\pi^{\oplus I}$. $\square$

The work above assumes that $\mathscr{A}$ is countable and that our continuous frames $\{\rho(\xi)f_j\}_{j\in I,\xi\in K}$ are taken over the measure space $I \times K$, where I is equipped with counting measure. We have imposed this assumption only for the sake of clarity. Our arguments work just as well (with obvious modifications) if we replace I with a σ-finite measure space (X, μ), and allow $\mathscr{A} = \{f_x\}_{x\in X}$ to be a possibly uncountable family of vectors. We have to assume, however, that the mapping $x \mapsto f_x$ is weakly measurable from X to $\mathcal{H}_\rho$. We also have to replace the direct sum $\mathcal{H}_\pi^{\oplus I}$ with the direct *integral* $\int_X^\oplus \mathcal{H}_\pi$. (See [26, §7.4] for a definition.) A standard measurability argument, which we omit, proves the mapping $X \times K \to \mathcal{H}_\rho$ given by $(x,\xi) \mapsto \rho(\xi)f_x$ is weakly measurable. We denote $E(\mathscr{A})$ for this mapping. As in the countable case, we write $f_x = (f_x^\pi)_{\pi\in\hat{K}}$ with $f_x^\pi = (f_{i,x}^\pi)_{i=1}^{m_\pi} \in \mathcal{H}_\pi^{\oplus m_\pi}$ and $f_{i,x}^\pi \in \mathcal{H}_\pi$. Strictly speaking,

$$\mathscr{A}(\pi) := (f_{i,x}^\pi)_{1\le i\le m_\pi, x\in X}$$

is no longer a matrix, but a sequence of mappings $X \to \mathcal{H}_\rho$, each given by $x \mapsto f_{i,x}^\pi$ for some i. For the sake of analogy, we will still call these mappings *rows* of $\mathscr{A}(\pi)$. Then we have the following results.

Theorem 4.6.8. *Let J be a range function in $\{\ell^2_{m_\pi}\}_{\pi\in\hat{K}}$, and assume that $\mathscr{A} \subseteq V_J$. For constants A and B with $0 < A \le B < \infty$, the following are equivalent.*

(i) $E(\mathscr{A})$ is a continuous frame for V_J with bounds A, B. That is,

$$A\|g\|^2 \le \int_X \int_K |\langle g, \rho(\xi)f_x\rangle|^2 \, d\xi \, dx \le B\|g\|^2 \qquad (g \in V_J).$$

(ii) For every $\pi \in \hat{K}$ and every sequence $(c_i)_{i=1}^{m_\pi} \in J(\pi) \subseteq \ell^2_{m_\pi}$,

$$d_\pi A \sum_{i=1}^{m_\pi} |c_i|^2 \leq \int_X \left\| \sum_{i=1}^{m_\pi} c_i f_{i,x}^\pi \right\|^2 dx \leq d_\pi B \sum_{i=1}^{m_\pi} |c_i|^2.$$

Corollary 4.6.9. *The following are equivalent for constants A and B with $0 < A \leq B < \infty$.*

(i) $E(\mathscr{A})$ is a continuous frame for $\mathcal{H}_\rho$ with bounds A, B.

(ii) For every $\pi \in \hat{K}$, the "rows" of $\mathscr{A}(\pi)$ belong to $\int_X^\oplus \mathcal{H}_\pi$, where they form a Riesz sequence with bounds $d_\pi A, d_\pi B$.

We end with an application. Remember that $\mathcal{M}_\pi \subseteq \mathcal{H}_\rho$ denotes the isotypical component of $\pi \in \hat{K}$ in ρ. In terms of our decomposition of $\mathcal{H}_\rho$, $\mathcal{M}_\pi$ is the summand $\mathcal{H}_\pi^{\oplus m_\pi} \subseteq \mathcal{H}_\rho$. We write P_π for orthogonal projection of $\mathcal{H}_\rho$ onto $\mathcal{M}_\pi$. The result below generalizes Corollary 4.5.7 for frames with multiple generators. It is a trivial consequence of Corollary 4.6.9.

Corollary 4.6.10. *Let $\mathscr{A}$ be as described in the paragraph above Theorem 4.6.8. The following are equivalent for constants A and B with $0 < A \leq B < \infty$.*

(i) $E(\mathscr{A})$ is a continuous frame for $\mathcal{H}_\rho$ with bounds A, B.

(ii) For each $\pi \in \hat{K}$, the mapping $X \times K \to \mathcal{M}_\pi$ given by $(x, \xi) \mapsto \rho(\xi) P_\pi f_x$ is a continuous frame for $\mathcal{M}_\pi$ with bounds A, B.

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